DISCRETE MATHS (Q 6 & 7, PAPER 2)

LESSON NO. 1: DIFFERENCE EQUATIONS

2006

6 (b) (i) Solve the difference equation $6u_{n+2} - 7u_{n+1} + u_n = 0$, where $n \geq 0$, given that $u_0 = 8$ and $u_1 = 3$.

(ii) Verify that the solution to part (**i**) also satisfies the difference equation $6u_{n+1} - u_n - 10 = 0$.

SOLUTION

6 (b) (i)

$u_n = l(\alpha)^n + m(\beta)^n$ **1**

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $6u_{n+2} - 7u_{n+1} + u_n = 0$

2. $6x^2 - 7x + 1 = 0$

3. $(x-1)(6x-1) = 0 \Rightarrow \alpha = 1,\ \beta = \frac{1}{6}$

4. $u_n = l(1)^n + m(\frac{1}{6})^n = l + m(\frac{1}{6})^n$

5. $u_0 = 8 = l + m$ and $u_1 = 3 = l + m(\frac{1}{6}) \Rightarrow 18 = 6l + m$

Solving these equations simultaneously $\Rightarrow l = 2,\ m = 6$

Ans: $u_n = 2 + 6(\frac{1}{6})^n = 2 + (\frac{1}{6})^{n-1}$

6 (b) (ii)

Substitute the solution into $6u_{n+1} - u_n - 10 = 0$.

$u_n = 2 + (\frac{1}{6})^{n-1}$

$u_{n+1} = 2 + (\frac{1}{6})^n$

$\Rightarrow 6u_{n+1} - u_n - 10 = 6(2 + (\frac{1}{6})^n) - (2 + (\frac{1}{6})^{n-1}) - 10$

$= 12 + (\frac{1}{6})^{n-1} - 2 - (\frac{1}{6})^{n-1} - 10 = 0$

2005

6 (b) (i) Solve the difference equation $u_{n+2} - 4u_{n+1} - 8u_n = 0$, where $n \geq 0$, given that $u_0 = 0$ and $u_1 = 2$.

(ii) Verify that your solution gives the correct value for u_2.

SOLUTION

6 (b) (i)

$u_n = l(\alpha)^n + m(\beta)^n$ **1**

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find *l*, *m* using extra conditions (boundary conditions).

1. $u_{n+2} - 4u_{n+1} - 8u_n = 0$

2. $x^2 - 4x - 8 = 0$

3. $x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(-8)}}{2} = \dfrac{4 \pm \sqrt{48}}{2} = \dfrac{4 \pm 4\sqrt{3}}{2} = 2 \pm 2\sqrt{3}$

$\Rightarrow \alpha = 2 + 2\sqrt{3},\ \beta = 2 - 2\sqrt{3}$

4. $u_n = l(2 + 2\sqrt{3})^n + m(2 - 2\sqrt{3})^n$

5. $u_0 = l + m = 0$ ....(**1**) and $u_1 = l(2 + 2\sqrt{3})^1 + m(2 - 2\sqrt{3})^1 = 2$ ....(**2**)

From equation (**1**) $\Rightarrow m = -l$

Substitute into equation (**2**): $\Rightarrow l(2 + 2\sqrt{3}) - l(2 - 2\sqrt{3}) = 2 \Rightarrow 2l + 2l\sqrt{3} - 2l + 2l\sqrt{3} = 2$

$\Rightarrow 4l\sqrt{3} = 2 \Rightarrow l = \dfrac{1}{2\sqrt{3}} = \dfrac{1}{2\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{6}$

$\therefore l = \dfrac{\sqrt{3}}{6},\ m = -\dfrac{\sqrt{3}}{6}$

Ans: $u_n = \frac{\sqrt{3}}{6}(2 + 2\sqrt{3})^n - \frac{\sqrt{3}}{6}(2 - 2\sqrt{3})^n$

6 (b) (ii)

Put *n* = 0 into the difference equation: $u_2 - 4u_1 - 8u_0 = 0 \Rightarrow u_2 = 4(2) + 8(0) = 8$

Put *n* = 2 into the solution:

$u_2 = \frac{\sqrt{3}}{6}(2 + 2\sqrt{3})^2 - \frac{\sqrt{3}}{6}(2 - 2\sqrt{3})^2 \Rightarrow u_2 = \frac{\sqrt{3}}{6}(4 + 8\sqrt{3} + 12 - 4 + 8\sqrt{3} - 12)$

$\Rightarrow u_2 = \frac{\sqrt{3}}{6}(4 + 8\sqrt{3} + 12 - 4 + 8\sqrt{3} - 12) = \frac{\sqrt{3}}{6}(16\sqrt{3}) = 8$

You have verified that you get the correct solution for u_2.

2004

6 (b) (i) Solve the difference equation $3u_{n+2} - 2u_{n+1} - u_n = 0$, where $n \geq 0$, given that $u_0 = 3$ and $u_1 = 7$.

(ii) Evaluate $\lim_{n \to \infty} u_n$.

SOLUTION

6 (b) (i)

$u_n = l(\alpha)^n + m(\beta)^n$ **1**

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $3u_{n+2} - 2u_{n+1} - u_n = 0$

2. $3x^2 - 2x - 1 = 0$

3. $\therefore (3x+1)(x-1) = 0 \Rightarrow \alpha = 1,\ \beta = -\frac{1}{3}$

4. $u_n = l(1)^n + m(-\frac{1}{3})^n = l + m(-\frac{1}{3})^n$

5. $u_0 = 3 = l + m$ and $u_1 = 7 = l + m(-\frac{1}{3}) \Rightarrow 21 = 3l - m$

Solving these equations simultaneously $\Rightarrow l = 6,\ m = -3$

Ans: $u_n = 6 - 3(-\frac{1}{3})^n$

6 (b) (ii)

You may be asked to find a limit to infinity.

Remember $\lim_{n \to \infty} r^n = 0,\ -1 < r < 1$ but $\lim_{n \to \infty} r^n = \infty,\ r < -1,\ r > 1$

$\lim_{n \to \infty} u_n = 6$

2003

6 (b) (i) Solve the difference equation $u_{n+2} - 4u_{n+1} + 3u_n = 0$, where $n \geq 0$, given that $u_o = -2$ and $u_1 = 4$.

(ii) Verify that the solution you have obtained in (**i**) satisfies the difference equation.

SOLUTION

6 (b) (i)

$u_n = l(\alpha)^n + m(\beta)^n$ **1**

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $u_{n+2} - 4u_{n+1} + 3u_n = 0$

2. $x^2 - 4x + 3 = 0$

3. $(x-1)(x-3) = 0 \Rightarrow \alpha = 1, \ \beta = 3$

4. $u_n = l(1)^n + m(3)^n = l + m(3)^n$

5. $u_0 = -2 = l + m$ and $u_1 = 4 = l + 3m$

Solving these equations simultaneously $\Rightarrow l = -5, \ m = 3$

Ans: $u_n = 3(3)^n - 5$

6 (b) (ii)

$u_n = 3(3)^n - 5$

$u_{n+1} = 3(3)^{n+1} - 5$

$u_{n+2} = 3(3)^{n+2} - 5$

$\Rightarrow 3(3)^{n+2} - 5 - 4[3(3)^{n+1} - 5] + 3[3(3)^n - 5] = 3(3)^{n+2} - 5 - 12(3)^{n+1} + 20 + 9(3)^n - 15$

$= (3)^n[3(3)^2 - 12(3) + 9] = (3)^n[27 - 36 + 9] = (3)^n[0] = 0$

2002

6 (b) (i) Solve the difference equation $6u_{n+2} - 5u_{n+1} + u_n = 0$, where $n \ge 0$, given that $u_0 = 5$ and $u_1 = 2$.

(ii) Find an expression in n for the sum of the terms $u_0 + u_1 + u_2 + ... + u_n$.
(Hint: It is the sum of two geometric series.)

(iii) Evaluate the sum to infinity of this series (that is: $\sum_{n=0}^{\infty} u_n$).

SOLUTION

6 (b) (i)

$u_n = l(\alpha)^n + m(\beta)^n$ 1

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $6u_{n+2} - 5u_{n+1} + u_n = 0$

2. $6x^2 - 5x + 1 = 0$

3. $(2x-1)(3x-1) = 0 \Rightarrow \alpha = \frac{1}{2}, \ \beta = \frac{1}{3}$

4. $u_n = l(\frac{1}{2})^n + m(\frac{1}{3})^n$

5. $u_0 = 5 = l + m$ and $u_1 = 2 = \frac{1}{2}l + \frac{1}{3}m \Rightarrow 12 = 3l + 2m$

Solving these equations simultaneously $\Rightarrow l = 2, \ m = 3$

Ans: $u_n = 2(\frac{1}{2})^n + 3(\frac{1}{3})^n$

6 (b) (ii)

Summing formula: $S_n = \dfrac{a(1-r^n)}{(1-r)}$ 5

$$u_n = 2(\tfrac{1}{2})^n + 3(\tfrac{1}{3})^n = 2[1 + (\tfrac{1}{2}) + (\tfrac{1}{2})^2 + ... + (\tfrac{1}{2})^{n+1}] + 3[1 + (\tfrac{1}{3}) + (\tfrac{1}{3})^2 + ... + (\tfrac{1}{3})^{n+1}]$$

↑ $a = 1, \ r = \frac{1}{2}, \ S_{n+1}$ ↑ $a = 1, \ r = \frac{1}{3}, \ S_{n+1}$

$$\Rightarrow u_n = \frac{2(1-(\frac{1}{2})^{n+1})}{1-\frac{1}{2}} + \frac{3(1-(\frac{1}{3})^{n+1})}{1-\frac{1}{3}} = 4(1-(\tfrac{1}{2})^{n+1}) + \tfrac{9}{2}(1-(\tfrac{1}{3})^{n+1})$$

6 (b) (iii)

$$\sum_{n=0}^{\infty} u_n = 4(1-0) + \tfrac{9}{2}(1-0) = \tfrac{17}{2}$$

FAMOUS LIMITS

$\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$. **Example**: $\lim_{n\to\infty} (\frac{3}{5})^n = 0$

$\lim_{n\to\infty} r^n = \infty$ for $r > 1, \ r < -1$. **Example**: $\lim_{n\to\infty} (\frac{3}{2})^n = \infty$

2001

6 (b) Solve the difference equation $u_{n+2} - 8u_{n+1} + 11u_n = 0$, where $n \geq 0$, given that $u_0 = 0$ and $u_1 = 2\sqrt{15}$.

SOLUTION

6 (b)

$u_n = l(\alpha)^n + m(\beta)^n$ 1

STEPS

1. Write the Second Order Difference Equation in decreasing order of subscripts: $pu_{n+2} + qu_{n+1} + ru_n = 0$
2. Write down the corresponding quadratic equation: $px^2 + qx + r = 0$
3. Solve this equation to find α, β.
4. Write solution as: $u_n = l(\alpha)^n + m(\beta)^n$
5. Find l, m using extra conditions (boundary conditions).

1. $u_{n+2} - 8u_{n+1} + 11u_n = 0$

2. $x^2 - 8x + 11 = 0$

3. $x = \dfrac{8 \pm \sqrt{64 - 4(1)(11)}}{2} = \dfrac{8 \pm \sqrt{20}}{2} = \dfrac{8 \pm 2\sqrt{5}}{2} = 4 \pm \sqrt{5}$

$\Rightarrow \alpha = 4 + \sqrt{5},\ \beta = 4 - \sqrt{5}$

4. $u_n = l(4+\sqrt{5})^n + m(4-\sqrt{5})^n$

5. $u_0 = l + m = 0 \Rightarrow m = -l$ and $u_1 = l(4+\sqrt{5}) + m(4-\sqrt{5}) = 2\sqrt{15}$

Solving these equations simultaneously:

$l(4+\sqrt{5}) - l(4-\sqrt{5}) = 2\sqrt{15} \Rightarrow 2l\sqrt{5} = 2\sqrt{15}$

$\Rightarrow l = \sqrt{3},\ m = -\sqrt{3}$

Ans: $u_n = \sqrt{3}(4+\sqrt{5})^n - \sqrt{3}(4-\sqrt{5})^n$