DISCRETE MATHS (Q 6 & 7, PAPER 2)

2007

6 (a) Six people, including Mary and John, sit in a row.

(i) How many different arrangements of the six people are possible.

(ii) In how many of these arrangements are Mary and John next to each other?

(b) α and β are the roots of the quadratic equation $px^2 + qx + r = 0$.

$u_n = l\alpha^n + m\beta^n$, for all $n \in \mathbf{N}$.

Show that $pu_{n+2} + qu_{n+1} + ru_n = 0$, for all $n \in \mathbf{N}$.

(c) w white discs and r red discs are placed in a box. Two of the discs are drawn at random from the box. The probability that both discs are red is p.

(i) Find p in terms of w and r.

(ii) When $w = 1$, find the value of r for which $p = \frac{1}{2}$.

(iii) There are other values of w and r that also give $p = \frac{1}{2}$.
The next smallest such value is even.
By investigating the even numbers in turn, find this value of w and the corresponding value of r.

SOLUTION

6 (a) (i)

Number of arrangements of six people

The number of arrangements of n different objects all taken, no repeats = $n!$ 8

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Number of ways = 6 × 5 × 4 × 3 × 2 × 1 = 6! = 720

6 (a) (ii)

Put Mary and John together and move them as a group. There are 5! ways to arrange these 4 people and the group of John and Mary. Within the group there are 2! ways to arrange John and Mary.

No. of arrangements = $5! \times 2! = 240$

6 (b)

REQUIRED TO PROVE: $u_n = l(\alpha)^n + m(\beta)^n$

PROOF

$\Rightarrow pu_{n+2} = pl(\alpha)^{n+2} + pm(\beta)^{n+2} \quad = pl\alpha^2(\alpha)^n + pm\beta^2(\beta)^n$

$\Rightarrow qu_{n+1} = ql(\alpha)^{n+1} + qm(\beta)^{n+1} \quad = ql\alpha(\alpha)^n + qm\beta(\beta)^n$

$\Rightarrow ru_n = rl(\alpha)^n + rm(\beta)^n \quad = rl(\alpha)^n + rm(\beta)^n$

$\Rightarrow pu_{n+2} + qu_{n+1} + ru_n = (\alpha)^n l(p\alpha^2 + q\alpha + r) + (\beta)^n m(p\beta^2 + q\beta + r)$

$= (\alpha)^n l(0) + (\beta)^n m(0) = 0 + 0 = 0$

once α, β are the roots of $px^2 + qx + r = 0$.

6 (c)

$$p(E) = \frac{\text{Number of desired outcomes}}{\text{Total possible number of outcomes}} = \frac{\#(E)}{\#(S)} \quad \ldots\ldots \quad 12$$

w white discs, *r* red discs. Total = (*r* + *w*) discs

6 (c) (i)

$$p(\text{Red, Red}) = \frac{r}{(r+w)} \times \frac{(r-1)}{(r+w-1)} = p$$

6 (c) (ii)

$$\Rightarrow \frac{r}{(r+1)} \times \frac{(r-1)}{(r+1-1)} = \frac{1}{2} \Rightarrow 2r(r-1) = r(r+1)$$

$$\Rightarrow 2r^2 - 2r = r^2 + r \Rightarrow r^2 - 3r = 0 \Rightarrow r(r-3) = 0$$

$$\Rightarrow r = 3$$

6 (c) (iii)

Try different even number values of *w*, solve for *r* until you get a solution for *r* that is a whole, positive number.
w = 2 and *w* = 4 do not work. *w* = 6 works.

$$\Rightarrow \frac{r}{(r+6)} \times \frac{(r-1)}{(r+5)} = \frac{1}{2} \Rightarrow 2r(r-1) = (r+6)(r+5)$$

$$\Rightarrow 2r^2 - 2r = r^2 + 11r + 30 \Rightarrow r^2 - 13r + 30 = 0$$

$$\Rightarrow (r-15)(r+2) = 0 \Rightarrow r = 15$$

7 (a) How many different selections of four letters can be made from the letters of the word FLORIDA?

(ii) How many of these selections contain at least one vowel?

(b) Two dice are thrown.

(i) What is the probability of getting two identical numbers or a total of five?

(ii) What is the probability that the product of the two numbers thrown is at least twice their sum?

(c) (i) Find, in terms of *a* and *d*, the mean of the first seven terms of an arithmetic sequence with first term *a* and common difference *d*.

(ii) Show that the standard deviation of these seven numbers is $2d$.

7 (a) (i)

The number of ways can you pick 4 objects can be picked from 7 objects: $^7C_4 = 35$

The number of selections of *n* different objects taking *r* at a time $= {}^nC_r$ 11

7 (a) (ii)

No. of selections with at least one vowel

= Total number of selections – Number of selections with no vowels

$= {}^7C_4 - {}^4C_4 = 35 - 1 = 34$

7 (b) (i)

	1	2	3	4	5	6
1	**(1, 1)**	(1, 2)	(1, 3)	**(1, 4)**	(1, 5)	(1, 6)
2	(2, 1)	**(2, 2)**	**(2, 3)**	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	**(3, 2)**	**(3, 3)**	(3, 4)	(3, 5)	(3, 6)
4	**(4, 1)**	(4, 2)	(4, 3)	**(4, 4)**	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	**(5, 5)**	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	**(6, 6)**

$$p(E) = \frac{\text{Number of desired outcomes}}{\text{Total possible number of outcomes}} = \frac{\#(E)}{\#(S)}$$ 12

$p(\text{Two identical numbers or a total of 5}) = \frac{10}{36} = \frac{5}{18}$

7 (b) (ii)

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	**(3, 6)**
4	(4, 1)	(4, 2)	(4, 3)	**(4, 4)**	**(4, 5)**	**(4, 6)**
5	(5, 1)	(5, 2)	(5, 3)	**(5, 4)**	**(5, 5)**	**(5, 6)**
6	(6, 1)	(6, 2)	**(6, 3)**	**(6, 4)**	**(6, 5)**	**(6, 6)**

$p(\text{Product of the two numbers is at least twice their sum}) = \frac{11}{36}$

7 (c) (i)

Terms: $a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ a+5d,\ a+6d$

$$\bar{x} = \frac{x_1 + x_2 + \ldots\ldots\ldots\ldots + x_N}{N} = \frac{\text{Sum of the Numbers}}{\text{Number of Numbers}} = \frac{\sum x}{N}$$ 3

Mean $\bar{x} = \dfrac{a+(a+d)+(a+2d)+(a+3d)+(a+4d)+(a+5d)+(a+6d)}{7}$

$$\Rightarrow \bar{x} = \frac{7a+21d}{7} = a+3d$$

7 (c) (ii)

$$\sigma = \sqrt{\frac{\text{Sum of (Deviations)}^2}{\text{Number of numbers}}} = \sqrt{\frac{\sum d^2}{N}}$$ 6

where deviation $d = (x - \bar{x}) = (\text{Number} - \text{Mean})$

x	d	d^2
a	$-3d$	$9d^2$
$a+d$	$-2d$	$4d^2$
$a+2d$	$-d$	d^2
$a+3d$	0	0
$a+4d$	d	d^2
$a+5d$	$2d$	$4d^2$
$a+6d$	$3d$	$9d^2$
		$28d^2$

$$\sigma = \sqrt{\frac{28d^2}{7}} = \sqrt{4d^2} = 2d$$