CALCULUS OPTION (Q 8, PAPER 2)

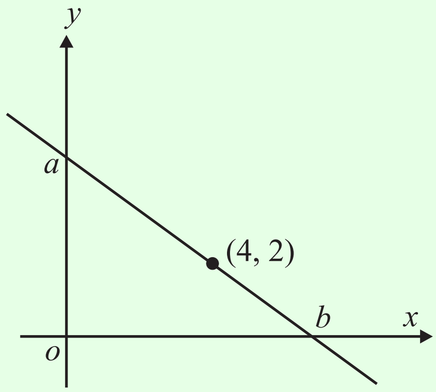
LESSON NO. 4: MAXIMISING AND MINIMISING FUNCTIONS

2006

8 (b) A line passes through the point (4, 2) and has slope m, where $m < 0$. The line intersects the axes at the points a and b.

(i) Find the co-ordinates of a and b, in terms of m.

(ii) Hence, find the value of m for which the area of triangle aob is a minimum.



SOLUTION

8 (b) (i)

Slope $= +\frac{m}{1}$

Equation of line L: $mx - y + k = 0$

$(4, 2) \in L \Rightarrow 4m - 2 + k = 0 \Rightarrow k = 2 - 4m$

Equation of line L: $mx - y + 2 - 4m = 0$

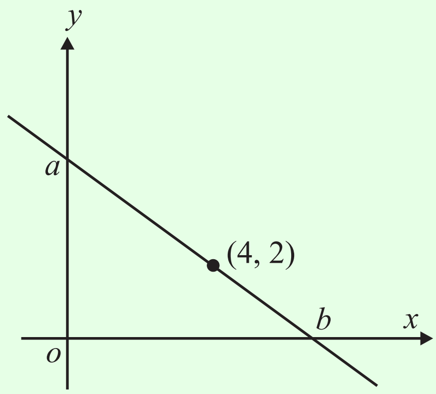
Y intercept: Put $x = 0 \Rightarrow 2 - 4m = y \ \therefore a(0, 2-4m)$

X intercept: Put $y = 0 \Rightarrow mx = 4m - 2 \Rightarrow x = \frac{4m-2}{m} \ \therefore b(\frac{4m-2}{m}, 0)$

8 (b) (ii)

STEPS
1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. A (Area)
2. Draw a diagram.



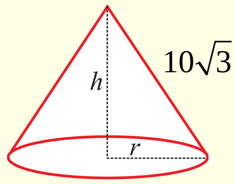
3. $A = \frac{1}{2}ba$
4. $a(0, 2-4m)$, $b(\frac{4m-2}{m}, 0)$
5. $A = \frac{1}{2}(\frac{4m-2}{m})(2-4m) = (2-\frac{1}{m})(2-4m) = 4 - 8m - \frac{2}{m} + 4$

   $\therefore A = 8 - 8m - \frac{2}{m} = 8 - 8m - 2m^{-1}$
6. $\frac{dA}{dm} = -8 + 2m^{-2} = 0 \Rightarrow \frac{2}{m^2} = 8 \Rightarrow m^2 = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$

   $\therefore m = -\frac{1}{2}$, $m < 0$

2005

8 (c) A cone has radius r cm, vertical height h cm and slant height $10\sqrt{3}$ cm. Find the value of h for which the volume is a maximum.



SOLUTION

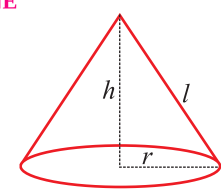
8 (c)

STEPS

1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

The cylinder formulae are found on page 6/7 of the tables as shown.

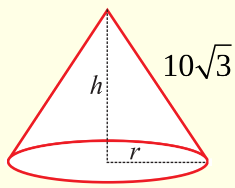
CONE



Curved surface area $= \pi r l$

Volume $= \frac{1}{3}\pi r^2 h$

1. V (Volume)
2. Draw a diagram.



3. $V = \frac{1}{3}\pi r^2 h$
4. $h^2 + r^2 = (10\sqrt{3})^2 \Rightarrow r^2 = 300 - h^2$ [Extra information]
5. $V = \frac{1}{3}\pi(300 - h^2)h = 100\pi h - \frac{1}{3}\pi h^3$
6. $\frac{dV}{dh} = 100\pi - \pi h^2 = 0 \Rightarrow 100 = h^2 \Rightarrow h = 10$ cm

Ans: $h = 10$ cm

2004

8 (c) A solid cylinder has height h and radius r. The height of the cylinder, added to the circumference of its base, is equal to 3 metres.

(i) Express the volume of the cylinder in terms of r and π.

(ii) Find the maximum possible volume of the cylinder in terms of π.

SOLUTION

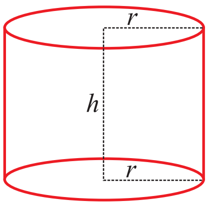
8 (c)

Use information from page 6/7 of the tables. The information you need is shown on the bottom of the page.

STEPS

1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

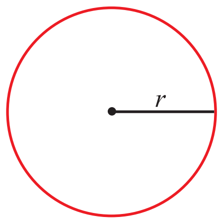
CYLINDER



Area of curved surface $= 2\pi rh$

Volume $= \pi r^2 h$

CIRCLE



Length $= 2\pi r$

Area $= \pi r^2$

1. V (Volume)
2. Draw a diagram.
3. $V = \pi r^2 h$
4. $h + 2\pi r = 3 \Rightarrow h = 3 - 2\pi r$ [Extra information]
5. $V = \pi r^2 h = \pi r^2(3 - 2\pi r) = 3\pi r^2 - 2\pi^2 r^3$
6. $\frac{dV}{dr} = 6\pi r - 6\pi^2 r^2 = 0 \Rightarrow 1 - \pi r = 0 \Rightarrow r = \frac{1}{\pi}$
7. $V_{\text{Max}} = 3\pi(\frac{1}{\pi})^2 - 2\pi^2(\frac{1}{\pi})^3 = \frac{3}{\pi} - \frac{2}{\pi} = \frac{1}{\pi}$

8 (c) (i)

$V = 3\pi r^2 - 2\pi^2 r^3$ [Step **5**]

8 (c) (ii)

$V_{\text{Max}} = \frac{1}{\pi}$ [Step **7**]

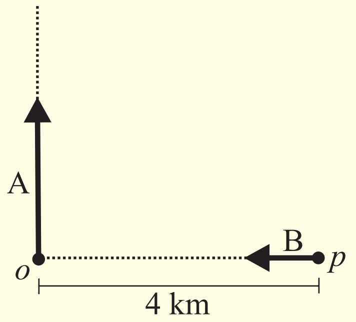
2003

8 (c) The point p is 4 km due east of the point o. At noon, A leaves o and travels north at a steady speed of 12 km/h. At the same time, B leaves p and travels towards o at a steady speed of 6 km/h.

(i) Write down expressions in x for the distances that A and B will each have travelled at x minutes after noon.

(ii) Find an expression in x for the distance that B will be from A at x minutes after noon.

(iii) At how many minutes after noon will B be closest to A?



SOLUTION

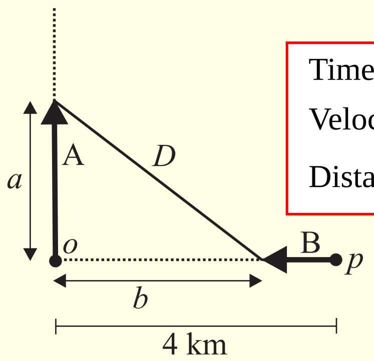
8 (c)

$$\text{Velocity} = \frac{\text{Distance}}{\text{Time}} \Rightarrow \text{Distance} = \text{Velocity} \quad \text{Time}$$

Time: x minutes

Velocity: 12 km/h $= \frac{1}{5}$ km/min

Distance: $\frac{x}{5}$ km



Time: x minutes

Velocity: 6 km/h $= \frac{1}{10}$ km/min

Distance: $\frac{x}{10}$ km

1. D (Distance apart)
2. Draw a diagram.
3. $D = \sqrt{a^2 + b^2}$
4. A: $a = \frac{x}{5}$

   B: $b = 4 - \frac{x}{10}$
5. $D = \sqrt{(\frac{x}{5})^2 + (4 - \frac{x}{10})^2} = [(\frac{x}{5})^2 + (4 - \frac{x}{10})^2]^{\frac{1}{2}}$
6. $\frac{dD}{dx} = \frac{1}{2}[(\frac{x}{5})^2 + (4 - \frac{x}{10})^2]^{-\frac{1}{2}}\{2(\frac{x}{5})(\frac{1}{5}) + 2(4 - \frac{x}{10})(-\frac{1}{10})\} = 0$

   $\Rightarrow \{2(\frac{x}{25}) + 2(4 - \frac{x}{10})(-\frac{1}{10})\} = 0 \Rightarrow \frac{x}{25} - \frac{2}{5} + \frac{x}{100} = 0$

   $\Rightarrow 4x - 40 + x = 0 \Rightarrow 5x = 40 \Rightarrow x = 8$ minutes

8 (c) (i) A: $\frac{x}{5}$ km B: $\frac{x}{10}$ km

8 (c) (ii) $D = \sqrt{(\frac{x}{5})^2 + (4 - \frac{x}{10})^2} = [(\frac{x}{5})^2 + (4 - \frac{x}{10})^2]^{\frac{1}{2}}$ [Step **5**]

8 (c) (iii) 8 minutes [Step **6**]

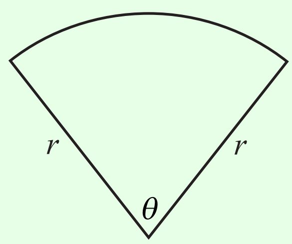
2002

8 (b) The perimeter of a sector of a circle of radius r is 8 metres.

(i) Express θ in terms of r, where θ is the angle of the sector in radians as shown.

(ii) Hence, show that the area of the sector, in square metres, is $4r - r^2$.

(iii) Find the maximum possible area of the sector.



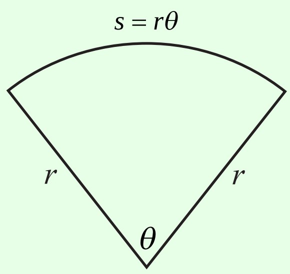
SOLUTION

8 (b)

> **STEPS**
> **1.** Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: *V* for volume.
> **2.** Draw a diagram (if necessary) and put in the variable(s).
> **3.** Write the quantity in terms of this/these variable(s).
> **4.** If there are 2 variables get rid of one in terms of the other using extra information.
> **5.** Hence, write the quantity as a function of a single variable.
> **6.** Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
> **7.** Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. A (Area)

2. Draw a diagram.



3. $A = \frac{1}{2}r^2\theta$

4. Perimeter = $2r + r\theta = 8$ (Extra Information)

$$\Rightarrow r\theta = 8 - 2r \Rightarrow \theta = \frac{8-2r}{r}$$

5. $A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{8-2r}{r}\right) = 4r - r^2$

6. $\frac{dA}{dr} = 4 - 2r = 0 \Rightarrow 4 = 2r \Rightarrow r = 2\,\text{m}$

7. $A_{\text{Max}} = 4r - r^2 = 4(2) - (2)^2 = 8 - 4 = 4\,\text{m}^2$

Arc length s

$s = r\theta$ **6**

Area A

$A = \frac{1}{2}r^2\theta$ **7**

8 (b) (i) $\theta = \frac{8-2r}{r}$ [Step **4**]

8 (b) (ii) Step **5**

8 (b) (iii) $A_{\text{Max}} = 4\,\text{m}^2$ [Step **7**]

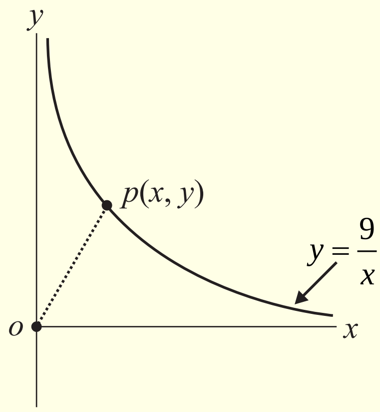
2001

8 (c) o is the origin, (0, 0). $p(x, y)$ is a point on the curve $y = \frac{9}{x}$, where $x > 0$. $|op|$ is the distance from the origin to p.

(i) Express $|op|$ in terms of x.

(ii) Given that there is one value of x for which $|op|$ is a minimum, find this value of x.

(iii) Hence, find the minimum value of $|op|$.



SOLUTION

8 (c)

STEPS

1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
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6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. D (Distance)

2. Draw a diagram.

3. $o(0, 0), p(x, y) \Rightarrow D = \sqrt{x^2 + y^2}$ (Distance formula)

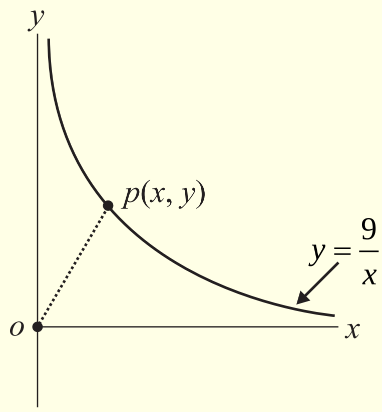
4. $y = \frac{9}{x}$ (Extra information)

5. $D = \sqrt{x^2 + \frac{81}{x^2}} = (x^2 + 81x^{-2})^{\frac{1}{2}}$

6. $\frac{dD}{dx} = \frac{1}{2}(x^2 + 81x^{-2})^{-\frac{1}{2}}[2x - 162x^{-3}] = 0$

$\Rightarrow 2x = \frac{162}{x^3} \Rightarrow x^4 = 81 \Rightarrow x = 3$

7. $D_{\text{Min.}} = \sqrt{9 + \frac{81}{9}} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$



8 (c) (i) $|op| = \sqrt{x^2 + \frac{81}{x^2}}$ [Step **3**]

8 (c) (ii) $x = 3$ [Step **6**]

8 (c) (iii) $3\sqrt{2}$ [Step **7**]