CALCULUS OPTION (Q 8, PAPER 2)

LESSON NO. 3: MACLAURIN SERIES

2006

8 (a) Derive the Maclaurin series for $f(x)=e^x$ up to and including the term containing x^3.

SOLUTION

8 (a) **THE MACLAURIN FORMULA**

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}...+\frac{f^{(n)}(0)x^n}{n!}+..$$ 3

$f(x)=e^x \Rightarrow f(0)=1$

$f'(x)=e^x \Rightarrow f'(0)=1$

$f''(x)=e^x \Rightarrow f''(0)=1$

$f'''(x)=e^x \Rightarrow f'''(0)=1$

$$e^x=\frac{1x^0}{0!}+\frac{1x^1}{1!}+\frac{1x^2}{2!}+\frac{1x^3}{3!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$$

2005

8 (b) (i) Derive the Maclaurin series for $f(x)=\ln(1+x)$ up to and including the term containing x^3.

(ii) Use those terms to find an approximation for $\ln\frac{11}{10}$.

(iii) Write down the general term of the series $f(x)$ and hence show that the series converges for $-1<x<1$.

SOLUTION

8 (b) (i) **THE MACLAURIN FORMULA**

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}...+\frac{f^{(n)}(0)x^n}{n!}+..$$ 3

$f(x)=\ln(1+x)\Rightarrow f(0)=0$

$f'(x)=\dfrac{1}{1+x}=(1+x)^{-1}\Rightarrow f'(0)=1$

$f''(x)=-1(1+x)^{-2}=-\dfrac{1}{(1+x)^2}\Rightarrow f''(0)=-1$

$f'''(x)=2(1+x)^{-3}=\dfrac{2}{(1+x)^3}\Rightarrow f'''(0)=2$

$$\therefore \ln(1+x)=\frac{0x^0}{0!}+\frac{1x^1}{1!}-\frac{1x^2}{2!}+\frac{2x^3}{3!}=x-\frac{x^2}{2}+\frac{x^3}{3}$$

Cont...

8 (b) (ii)

$\ln \frac{11}{10} = \ln(1+\frac{1}{10})$

Replace x by $\frac{1}{10}$ in the series formula.

$\therefore \ln \frac{11}{10} = (\frac{1}{10}) - \frac{1}{2}(\frac{1}{10})^2 + \frac{1}{3}(\frac{1}{10})^3 = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} = \frac{143}{1500}$

8 (b) (iii)

STEPS TO FIND GENERAL TERM

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series T_n to generate u_n.

   $T_n = a + (n-1)d$ **4**

2. Sometimes the signs alternate: +, −, +, −, +, −....... Multiply by $(-1)^{n-1}$ to achieve this alternation.

Powers, Factorial: 1, 2, 3,..... [Arithmetic series a = 1, d = 1]

$T_n = 1 + (n-1)1 = n$

Signs alternate.

Therefore, general term for ln (1 + x): $u_n = (-1)^{n-1}\dfrac{x^n}{n}$

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. **2**

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.
2. Find u_{n+1}.
3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = (-1)^{n-1}\dfrac{x^n}{n}$

2. $u_{n+1} = (-1)^n \dfrac{x^{n+1}}{(n+1)}$

3. $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{(-1)^n x^{n+1}}{(n+1)} \times \dfrac{n}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\left|(-1)^1 x \dfrac{n}{(n+1)}\right| = \lim_{n\to\infty}\left|(-1)^1 x \dfrac{n}{n(1+\frac{1}{n})}\right| = |x|$

The series is convergent if $|x| < 1 \Rightarrow -1 < x < 1$

2004

8 (b) $f(x)=f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+..$ is the Maclaurin series.

(i) Derive the first five terms of the Maclaurin series for e^x.

(ii) Hence write down the first five terms of the Maclaurin series for e^{-x} and deduce the first three non-zero terms of the series for $\frac{e^x+e^{-x}}{2}$.

(iii) Write the general term of the series for $\frac{e^x+e^{-x}}{2}$ and use the Ratio Test to show that the series converges for all x.

SOLUTION

8 (b) (i) **THE MACLAURIN FORMULA**

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}...+\frac{f^{(n)}(0)x^n}{n!}+..$$ **3**

$f(x)=e^x \Rightarrow f(0)=1$

$f'(x)=e^x \Rightarrow f'(0)=1$

$f''(x)=e^x \Rightarrow f''(0)=1$

$f'''(x)=e^x \Rightarrow f'''(0)=1$

$f^{iv}(x)=e^x \Rightarrow f^{iv}(0)=1$

$$e^x=\frac{1x^0}{0!}+\frac{1x^1}{1!}+\frac{1x^2}{2!}+\frac{1x^3}{3!}+\frac{1x^4}{4!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$

8 (b) (ii)

Replace x by $-x$.

$$e^{-x}=1+\frac{(-x)}{1!}+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}$$

$$\frac{e^x+e^{-x}}{2}=\frac{1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}}{2}$$

$$=\frac{1+\frac{x^2}{2!}+\frac{x^4}{4!}+1+\frac{x^2}{2!}+\frac{x^4}{4!}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}$$

Cont...

8 (b) (iii)

8 (b) (i)

Steps to find general term

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series T_n to generate u_n.

   $T_n = a + (n-1)d$ 4

2. Sometimes the signs alternate: $+, -, +, -, +, -$....... Multiply by $(-1)^{n-1}$ to achieve this alternation.

Powers, Factorial: 0, 2, 4,..... [Arithmetic series a = 0, d = 2]

$T_n = 0 + (n-1)2 = 2n - 2$

Therefore, general term for e^x: $u_n = \dfrac{x^{2n-2}}{(2n-2)!}$

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| > 1$. 2

8 (b) (ii)

Steps

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.
2. Find u_{n+1}.
3. Evaluate $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = \dfrac{x^{2n-2}}{(2n-2)!}$

2. $u_{n+1} = \dfrac{x^{2n}}{(2n)!}$

3. $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{x^{2n}}{(2n)!} \times \dfrac{(2n-2)!}{x^{2n-2}}\right| = \lim_{n\to\infty}\left|\dfrac{x^2}{(2n)(2n-1)}\right| = 0 < 1$

Therefore, this series is convergent for all values of x.

2003

8 (b) $f(x)=f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+..$ is the Maclaurin series.

(i) Derive the Maclaurin series for $f(x)=\log_e(1+x)$ up to an including the term containing x^4.

(ii) Write down the general term and use the Ratio Test to show that the series converges for $-1<x<1$.

Solution

8 (b) **The Maclaurin Formula**

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}...+\frac{f^{(n)}(0)x^n}{n!}+..$$ 3

8 (b) (i)

$f(x)=\ln(1+x)\Rightarrow f(0)=0$

$f'(x)=\frac{1}{1+x}=(1+x)^{-1}\Rightarrow f'(0)=1$

$f''(x)=-1(1+x)^{-2}=-\frac{1}{(1+x)^2}\Rightarrow f''(0)=-1$

$f'''(x)=2(1+x)^{-3}=\frac{2}{(1+x)^3}\Rightarrow f'''(0)=2$

$f^{iv}(x)=-6(1+x)^{-4}\Rightarrow f^{iv}(0)=-6$

$$f(x)=\frac{0x^0}{0!}+\frac{1x^1}{1!}-\frac{1x^2}{2!}+\frac{2x^3}{3!}-\frac{6x^4}{4!}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$$

8 (b) (ii)

Steps to find general term

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series T_n to generate u_n.

$T_n=a+(n-1)d$ 4

2. Sometimes the signs alternate: $+, -, +, -, +, -$....... Multiply by $(-1)^{n-1}$ to achieve this alternation.

Powers, Factorial: 1, 2, 3,..... [Arithmetic series a = 1, d = 1]

$T_n=1+(n-1)1=n$

Signs alternate.

Therefore, general term for ln (1 + x): $u_n=(-1)^{n-1}\frac{x^n}{n}$

Cont...

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. 2

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = (-1)^{n-1}\frac{x^n}{n}$

2. $u_{n+1} = (-1)^n \frac{x^{n+1}}{(n+1)}$

3. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)} \times \frac{n}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\left|(-1)^1 x \frac{n}{(n+1)}\right| = \lim_{n\to\infty}\left|(-1)^1 x \frac{n}{n(1+\frac{1}{n})}\right| = |x|$

The series is convergent if $|x| < 1 \Rightarrow -1 < x < 1$

2002

8 (c) The Maclaurin series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ The series is convergent when $|x| < 1$.

(i) Write down the first four terms in the series expansion for $\tan^{-1} \frac{1}{2}$.

(ii) Use the fact that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}$ to derive a series expansion for π, giving the terms up to and including seventh powers.

(iii) Use these terms to find an approximation for π. Give your answer correct to four places of decimal.

SOLUTION

8 (c)

The Maclaurin series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ The series is convergent when $|x| < 1$.

8 (c) (i)

$$\tan^{-1} \tfrac{1}{2} = (\tfrac{1}{2}) - \frac{(\frac{1}{2})^3}{3} + \frac{(\frac{1}{2})^5}{5} - \frac{(\frac{1}{2})^7}{7} = \tfrac{1}{2} - \tfrac{1}{24} + \tfrac{1}{160} - \tfrac{1}{896}$$

8 (c) (ii)

$$\tan^{-1} \tfrac{1}{3} = (\tfrac{1}{3}) - \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} - \frac{(\frac{1}{3})^7}{7} = \tfrac{1}{3} - \tfrac{1}{81} + \tfrac{1}{1215} - \tfrac{1}{15309}$$

$$\therefore \pi = 4[\tan^{-1} \tfrac{1}{2} + \tan^{-1} \tfrac{1}{3}] = 4[\tfrac{1}{2} - \tfrac{1}{24} + \tfrac{1}{160} - \tfrac{1}{896} + \tfrac{1}{3} - \tfrac{1}{81} + \tfrac{1}{1215} - \tfrac{1}{15309}]$$

8 (c) (iii)

$\therefore \pi = 3 \cdot 1409$

2001

8 (b) $f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + ..$ is the Maclaurin series for $f(x)$.

(i) Derive the Maclaurin series for $f(x) = \sin x$ uo to and including the term containing x^7.

(ii) Write down the general term and use the Ratio Test to show that the series converges for all $x \in \mathbf{R}$.

SOLUTION

8 (b) **THE MACLAURIN FORMULA**

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} ... + \frac{f^{(n)}(0)x^n}{n!} + ..$$ 3

8 (b) (i)

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(0) = 0$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = 1$

$f^{vi}(x) = -\sin x \Rightarrow f^{vi}(0) = 0$

$f^{vii}(x) = -\cos x \Rightarrow f^{vii}(0) = -1$

$$\Rightarrow \sin x = \frac{0x^0}{0!} + \frac{1x^1}{1!} + \frac{0x^2}{2!} - \frac{1x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} + \frac{0x^6}{6!} - \frac{1x^7}{7!}$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

8 (b) (ii)

STEPS TO FIND GENERAL TERM

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series T_n to generate u_n.

   $T_n = a + (n-1)d$ 4

2. Sometimes the signs alternate: +, −, +, −, +, −....... Multiply by $(-1)^{n-1}$ to achieve this alternation.

Powers, Factorial: 1, 3, 5,..... [Arithmetic series a = 1, d = 2]

$T_n = 1 + (n-1)2 = 2n - 1$

Signs alternate.

Therefore, general term for sin x: $u_n = (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$

Cont...

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. 2

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = (-1)^{n-1}\dfrac{x^{2n-1}}{(2n-1)!}$

2. $u_{n+1} = (-1)^n\dfrac{x^{2n+1}}{(2n+1)!}$

3. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\times\frac{(2n-1)!}{(-1)^{n-1}x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^1 x^2}{(2n+1)2n}\right| = 0 < 1$

Therefore, it is convergent.