CALCULUS OPTION (Q 8, PAPER 2)

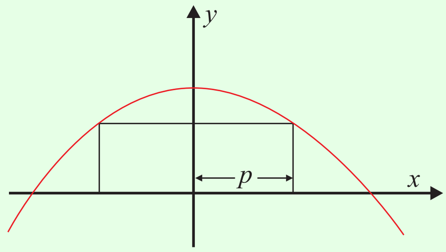
2010

8 (a) Use integration by parts to find $\int \log_e x \, dx$.

(b) A rectangle is inscribed between the curve $y = 9 - x^2$ and the x-axis, as shown.

(i) Write an expression for the area of the rectangle in terms of p.

(ii) Hence, calculate the area of the largest possible rectangle.



(c) (i) Derive the Maclaurin series for $f(x) = \cos x$ up to and including the term containing x^6.

(ii) Hence, and using the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, show that the first three non-zero terms of the Maclaurin series for $\sin^2 x$ are $x^2 - \frac{x^4}{3} + \frac{2x^6}{45}$.

(iii) Use these terms to find an approximation for $\sin^2(\frac{1}{2})$, as a fraction.

SOLUTION

8 (a)

$$\int u \, dv = uv - \int v \, du$$

STEPS

1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation; Let dv equal what is left and find v by integration.
   NOTE: **LIATE** helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v \, du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

LIST of Functions
1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1. $I = \int (\log_e x) \times 1 \, dx$

2.

$u = \log_e x$	$dv = 1dx$
$du = \frac{1}{x} dx$	$v = x$

3. $I = (\log_e x)x - \int x \times \frac{1}{x} dx$

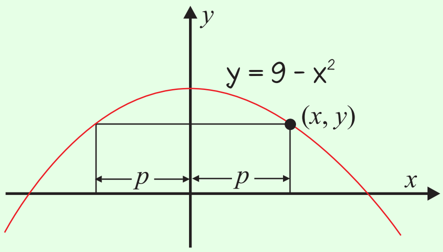
$= x \log_e x - \int 1 dx$

$= x \log_e x - x + c$

8 (b) (i)

Steps
1. Identify the quantity to be maximised/minimised and give it a suitable symbol. **Example**: *V* for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. Area of rectangle, A
2. Diagram shown
3. $A = 2py$
4. Extra information: $x = p \Rightarrow y = 9 - p^2$
5. $A = 2py = 2p(9-p^2) = 18p - 2p^3$



6. $\frac{dA}{dp} = 0 \Rightarrow 18p - 2p^3 = 0$

$$9 - p^2 = 0$$
$$9 = p^2$$
$$\therefore p = 3$$

7. $A_{\text{Max}} = 18(\sqrt{3}) - 2(\sqrt{3})^3 = 18(\sqrt{3}) - 6(\sqrt{3}) = 12\sqrt{3}$

Answers: (i) $A = 18p - 2p^3$

(ii) $A_{\text{Max}} = 12\sqrt{3}$

8 (c) (i)

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \ldots + \frac{f^{(n)}(0)x^n}{n!} + ..$$

$$f(x) = \cos x \Rightarrow f(0) = 1$$
$$f'(x) = -\sin x \Rightarrow f'(0) = 0$$
$$f''(x) = -\cos x \Rightarrow f''(0) = -1$$
$$f'''(x) = \sin x \Rightarrow f'''(0) = 0$$
$$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = 1$$
$$f^{v}(x) = -\sin x \Rightarrow f^{v}(0) = 0$$
$$f^{v'}(x) = -\cos x \Rightarrow f^{v'}(0) = -1$$

$$\cos x = \frac{1x^0}{0!} + \frac{0x^1}{1!} - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!}$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

8 (c) (ii)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!}$$

$$= 1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \frac{64x^6}{720}$$

$$= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45}$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$$= \tfrac{1}{2}\left(1 - 1 + 2x^2 - \frac{2x^4}{3} + \frac{4x^6}{45}\right)$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45}$$

8 (c) (iii)

$$\sin^2(\tfrac{1}{2}) = (\tfrac{1}{2})^2 - \frac{(\tfrac{1}{2})^4}{3} + \frac{2(\tfrac{1}{2})^6}{45} = \frac{331}{1440}$$