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CALCULUS OPTION (Q 8, PAPER 2)

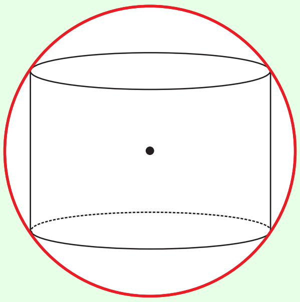
2009

8 (a) Use integration by parts to find $\int xe^{4x}dx$.

(b) (i) Derive the first four terms of the Maclaurin series for $f(x)=\sqrt{1+x}$.

(ii) Given that this series converges for $-1 < x < 1$, use these four terms to find an approximation for $\sqrt{17}$, as a fraction.

(c) The diagram shows a cylinder inscribed in a sphere.
The cylinder has height $2x$ and radius r.
The sphere has fixed radius a.



(i) Express r in terms of a and x.

(ii) Find, in terms of a, the maximum possible volume of the cylinder.

SOLUTION

8 (a)

$$\int u\,dv = uv-\int v\,du$$

STEPS

1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation;
   Let dv equal what is left and find v by integration.
   NOTE: LIATE helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v\,du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

LIST of Functions
1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1. $I=\int xe^{4x}dx$

2.

$$u=x \qquad dv=e^{4x}dx$$
$$du=dx \qquad v=\tfrac{1}{4}e^{4x}$$

3. $I=x(\tfrac{1}{4}e^{4x})-\int(\tfrac{1}{4}e^{4x})\,dx$

$=\tfrac{1}{4}xe^{4x}-\tfrac{1}{4}\int e^{4x}\,dx$

$=\tfrac{1}{4}xe^{4x}-\tfrac{1}{4}\times\tfrac{1}{4}e^{4x}+c$

$=\tfrac{1}{4}xe^{4x}-\tfrac{1}{16}e^{4x}+c$

$=\tfrac{1}{16}e^{4x}(4x-1)+c$

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8 (b) (i)

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}\ldots+\frac{f^{(n)}(0)x^n}{n!}+..$$

$$f(x)=\sqrt{1+x}=(1+x)^{\frac{1}{2}}\Rightarrow f(0)=1$$

$$f'(x)=\tfrac{1}{2}(1+x)^{-\frac{1}{2}}(1)=\frac{1}{2\sqrt{1+x}}\Rightarrow f'(0)=\tfrac{1}{2}$$

$$f''(x)=-\tfrac{1}{4}(1+x)^{-\frac{3}{2}}=-\frac{1}{4(1+x)^{\frac{3}{2}}}\Rightarrow f''(0)=-\tfrac{1}{4}$$

$$f'''(x)=\tfrac{3}{8}(1+x)^{-\frac{5}{2}}=\frac{3}{8(1+x)^{\frac{5}{2}}}\Rightarrow f'''(0)=\tfrac{3}{8}$$

$$\therefore\ f(x)=\sqrt{1+x}=\frac{1x^0}{0!}+\frac{\frac{1}{2}x^1}{1!}-\frac{\frac{1}{4}x^2}{2!}+\frac{\frac{3}{8}x^3}{3!}\ldots\ldots..$$

$$=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3\ldots\ldots.$$

8 (b) (ii)

$$\sqrt{17}=\sqrt{1+16}=\sqrt{16(\tfrac{1}{16}+1)}=4\sqrt{1+\tfrac{1}{16}}$$

$$x=\tfrac{1}{16}\,(-1<x<1)$$

$$\therefore\ \sqrt{17}=1+\tfrac{1}{2}(\tfrac{1}{16})-\tfrac{1}{8}(\tfrac{1}{16})^2+\tfrac{1}{16}(\tfrac{1}{16})^3=\tfrac{67553}{16384}$$

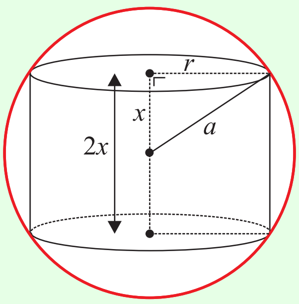
8 (c) (i)

Steps

1. Identify the quantity to be maximised/minimised and give it a suitable symbol. **Example**: *V* for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. Volume of cylinder, *V*

2. Diagram shown.



3. $V=\pi r^2h=\pi r^2(2x)=2\pi r^2x$

4. Extra Information: $a^2=r^2+x^2\Rightarrow r=\sqrt{a^2-x^2}$

5. $V=2\pi(a^2-x^2)x=2\pi a^2x-2\pi x^3$

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6. $\frac{dV}{dx} = 0 \Rightarrow 2\pi a^2 - 6\pi x^2 = 0$

$$a^2 - 3x^2 = 0$$

$$a^2 = 3x^2$$

$$\therefore x = \frac{a}{\sqrt{3}}$$

7. $V_{\text{Max}} = 2\pi a^2\left(\frac{a}{\sqrt{3}}\right) - 2\pi\left(\frac{a}{\sqrt{3}}\right)^3$

$$= \frac{2\pi a^3}{\sqrt{3}} - \frac{2\pi a^3}{3\sqrt{3}} = \frac{4\pi a^3}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{4\sqrt{3}\pi a^3}{9}$$

ANSWERS: (i) $r = \sqrt{a^2 - x^2}$

(ii) $V_{\text{Max}} = \frac{4}{9}\sqrt{3}\pi a^3$