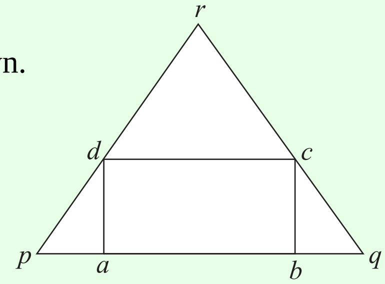
CALCULUS OPTION (Q 8, PAPER 2)

2008

8 (a) Use the ratio test to show that $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n!}$ is convergent.

(b) *pqr* is an equilateral triangle of side 6 cm.
abcd is a rectangle inscribed in the triangle as shown.
$|ab| = x$ cm and $|bc| = y$ cm.



(i) Express *y* in terms of *x*.

(ii) Find the maximum possible area of *abcd*.

(c) (i) Derive the Maclaurin series for $f(x) = \cos x$, up to and including the term containing x^4.

(ii) Hence, or otherwise, show that the first three non-zero terms of the Maclaurin series for $f(x) = \cos^2 x$ are $1 - x^2 + \frac{x^4}{3}$.

(iii) Use these to find an approximation for $\cos^2(0 \cdot 2)$, giving your answer correct to four decimal places.

SOLUTION

8 (a)

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. 2

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = \frac{2^{3n+1}}{n!}$

2. $u_{n+1} = \frac{2^{3n+4}}{(n+1)!}$

3. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2^{3n+4}}{(n+1)!} \times \frac{n!}{2^{3n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^3}{n+1}\right| = 0 < 1$

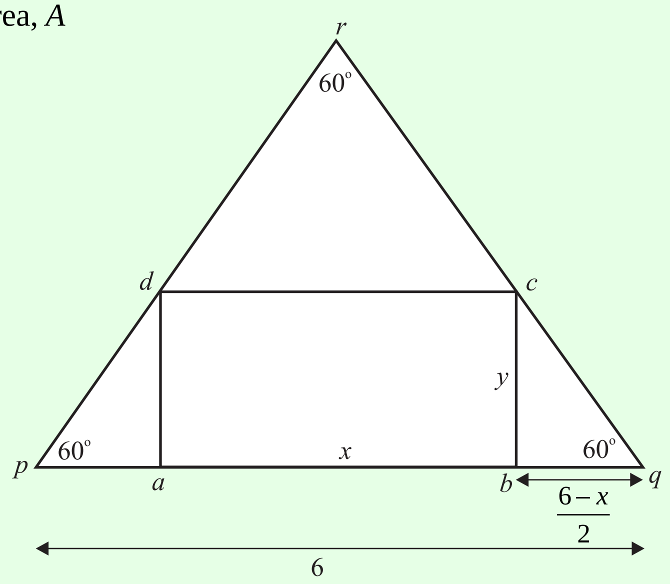
8 (b)

Steps

1. Identify the quantity to be maximised/minimised and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. Area, A

2.



3. $A = xy$

4. Extra information: $\tan 60^\circ = \dfrac{y}{\frac{6-x}{2}} \Rightarrow \sqrt{3} = \dfrac{2y}{6-x}$

$$\Rightarrow 2y = \sqrt{3}(6-x)$$

$$\therefore y = \sqrt{3}(3 - \tfrac{x}{2})$$

5. $A = xy = \sqrt{3}x(3 - \tfrac{x}{2})$

$$\therefore A = 3\sqrt{3}x - \tfrac{\sqrt{3}}{2}x^2$$

6. $\dfrac{dA}{dx} = 0 \Rightarrow \dfrac{dA}{dx} = 3\sqrt{3} - \sqrt{3}x = 0$

$$\Rightarrow 3\sqrt{3} = \sqrt{3}x$$

$$\therefore x = 3$$

7. $A_{\text{Max.}} = 3\sqrt{3}(3) - \tfrac{\sqrt{3}}{2}(3)^2 = 9\sqrt{3} - \tfrac{9}{2}\sqrt{3}$

$$\therefore A_{\text{Max.}} = \tfrac{9}{2}\sqrt{3}$$

Answers

8 (b) (i) $x = 3$

8 (b) (ii) $A_{\text{Max.}} = \tfrac{9}{2}\sqrt{3}$

8 (c) (i)

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}...+\frac{f^{(n)}(0)x^n}{n!}+..$$ 3

$f(x)=\cos x \Rightarrow f(0)=1$

$f'(x)=-\sin x \Rightarrow f'(0)=0$

$f''(x)=-\cos x \Rightarrow f''(0)=-1$

$f'''(x)=\sin x \Rightarrow f'''(0)=0$

$f'^{v}(x)=\cos x \Rightarrow f'^{v}(0)=1$

$$\therefore \cos x=\frac{1x^0}{0!}-\frac{1x^2}{2!}+\frac{1x^4}{4!}=1-\frac{x^2}{2}+\frac{x^4}{24}$$

8 (c) (ii)

$$\therefore \cos^2 x=\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)$$

$$=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^2}{2}+\frac{x^4}{4}-\frac{x^6}{48}+\frac{x^4}{24}-\frac{x^6}{48}....$$

$$=1-x^2+\frac{x^4}{3}$$

OR

$$\cos^2 x=\tfrac{1}{2}(1+\cos 2x)$$

$$\Rightarrow \cos^2 x=\frac{1}{2}\left(1+1-\frac{(2x)^2}{2}+\frac{(2x)^4}{24}\right)$$

$$\Rightarrow \cos^2 x=\frac{1}{2}\left(2-\frac{4x^2}{2}+\frac{16x^4}{24}\right)=\frac{1}{2}\left(2-2x^2+\frac{2x^4}{3}\right)$$

$$\therefore \cos^2 x=1-x^2+\frac{x^4}{3}$$

8 (c) (iii)

$$\therefore \cos^2(0.2)=1-(0.2)^2+\frac{(0.2)^4}{3}=0.9605 \quad \text{[Use calculator]}$$