Calculus Option (Q 8, Paper 2)

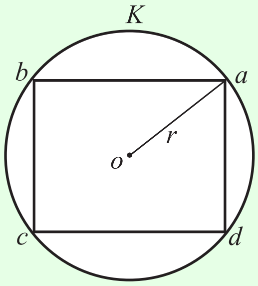
2000

8 (a) Use the ratio test to show that $\sum_{n=1}^{\infty} \frac{(n+2)!}{2^{n+2}}$ is divergent.

8 (b) (i) Use integration by parts to find $\int e^{2x} \cos x \, dx$.

(ii) Given that $\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx = \frac{1}{n}(e^{\pi} - 2)$, find the value of n where $n \in \mathbf{N}$.

8 (c) K is a circle with centre o.
a, b, c and d are points on K such that $abcd$ is a rectangle.
$|oa| = r$ cm; $|ab| = 2x$ cm and $|ad| = 2y$ cm.



(i) Express y in terms of x and r.

(ii) Hence, or otherwise, show that the maximum area of $abcd$ is $2r^2$ cm^2.

Solution

8 (a)

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. **2**

Steps

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.
2. Find u_{n+1}.
3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = \frac{(n+2)!}{2^{n+2}}$

2. $u_{n+1} = \frac{(n+3)!}{2^{n+3}}$

3. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+3)!}{2^{n+3}} \times \frac{2^{n+2}}{(n+2)!}\right| = \lim_{n\to\infty}\left|\frac{n+3}{2}\right| = \infty > 1$

$\Rightarrow$ Series is divergent.

8 (b) (i)

PARTS FORMULA $\int u\,dv = uv - \int v\,du$ .... ❶

STEPS

1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation;
   Let dv equal what is left and find v by integration.
   NOTE: **LIATE** helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v\,du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

LIST of Functions
1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1. $I = \int e^{2x} \cos x\,dx$

2.

$$u = \cos x \qquad dv = e^{2x}dx$$
$$du = -\sin x dx \qquad v = \tfrac{1}{2}e^{2x}$$

3. $\therefore I = (\cos x)(\tfrac{1}{2}e^{2x}) - \int \tfrac{1}{2}e^{2x}(-\sin x\,dx)$

$\Rightarrow I = \tfrac{1}{2}e^{2x}\cos x + \tfrac{1}{2}\int e^{2x}\sin x\,dx$

$$u = \sin x \qquad dv = e^{2x}dx$$
$$du = \cos x\,dx \qquad v = \tfrac{1}{2}e^{2x}$$

$\Rightarrow I = \tfrac{1}{2}e^{2x}\cos x + \tfrac{1}{2}\{(\sin x)(\tfrac{1}{2}e^{2x}) - \int(\tfrac{1}{2}e^{2x})\cos x\,dx\}$

$\Rightarrow I = \tfrac{1}{2}e^{2x}\cos x + \tfrac{1}{4}e^{2x}\sin x - \tfrac{1}{4}\int e^{2x}\cos x\,dx$

$\Rightarrow I = \tfrac{1}{2}e^{2x}\cos x + \tfrac{1}{4}e^{2x}\sin x - \tfrac{1}{4}I + c$

$\Rightarrow \tfrac{5}{4}I = \tfrac{1}{2}e^{2x}\cos x + \tfrac{1}{4}e^{2x}\sin x + c$

$\Rightarrow I = \tfrac{4}{5}(\tfrac{1}{2}e^{2x}\cos x + \tfrac{1}{4}e^{2x}\sin x) + c$

$\therefore I = \tfrac{2}{5}e^{2x}\cos x + \tfrac{1}{5}e^{2x}\sin x + c$

8 (b) (i)

$\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \tfrac{1}{n}(e^{\pi} - 2)$

$\Rightarrow [\tfrac{2}{5}e^{2x}\cos x + \tfrac{1}{5}e^{2x}\sin x]_0^{\frac{\pi}{2}} = \tfrac{1}{n}(e^{\pi} - 2)$

$\Rightarrow [(\tfrac{2}{5}e^{\pi}\cos\tfrac{\pi}{2} + \tfrac{1}{5}e^{\pi}\sin\tfrac{\pi}{2}) - (\tfrac{2}{5}e^{0}\cos 0 + \tfrac{1}{5}e^{0}\sin 0)] = \tfrac{1}{n}(e^{\pi} - 2)$

$\Rightarrow [(0 + \tfrac{1}{5}e^{\pi}) - (\tfrac{2}{5})] = \tfrac{1}{n}(e^{\pi} - 2)$

$\Rightarrow \tfrac{1}{5}e^{\pi} - \tfrac{2}{5} = \tfrac{1}{n}(e^{\pi} - 2)$

$\Rightarrow \tfrac{1}{5}(e^{\pi} - 2) = \tfrac{1}{n}(e^{\pi} - 2)$

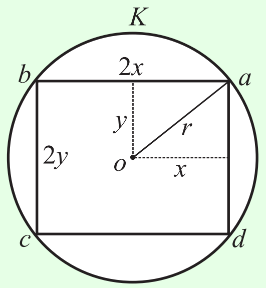
$\therefore n = 5$

8 (c)

Steps
1. Identify the quantity to be maximised/minimised and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. Area, A

2. Diagram shown.



3. $A = 4xy$

4. $x^2 + y^2 = r^2 \Rightarrow y = \sqrt{r^2 - x^2}$

5. $\therefore A = 4x\sqrt{r^2 - x^2}$

6. $\dfrac{dA}{dx} = 4x[\tfrac{1}{2}(r^2 - x^2)^{-\frac{1}{2}}(-2x)] + (r^2 - x^2)^{\frac{1}{2}}(4)$

$\Rightarrow \dfrac{dA}{dx} = -4x^2(r^2 - x^2)^{-\frac{1}{2}} + 4(r^2 - x^2)^{\frac{1}{2}}$ [Multiply above and below by $(r^2 - x^2)^{\frac{1}{2}}$.]

$$\Rightarrow \frac{dA}{dx} = \frac{[-4x^2(r^2 - x^2)^{-\frac{1}{2}} + 4(r^2 - x^2)^{\frac{1}{2}}]}{1}\frac{(r^2 - x^2)^{\frac{1}{2}}}{(r^2 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dA}{dx} = \frac{[-4x^2 + 4(r^2 - x^2)]}{(r^2 - x^2)^{\frac{1}{2}}} = \frac{[-4x^2 + 4r^2 - 4x^2]}{(r^2 - x^2)^{\frac{1}{2}}}$$

$$\therefore \frac{dA}{dx} = \frac{4r^2 - 8x^2}{(r^2 - x^2)^{\frac{1}{2}}}$$

$$\frac{dA}{dx} = 0 \Rightarrow \frac{4r^2 - 8x^2}{(r^2 - x^2)^{\frac{1}{2}}} = 0$$

$\Rightarrow 4r^2 - 8x^2 = 0$

$\Rightarrow r^2 = 2x^2$

$\therefore x = \dfrac{r}{\sqrt{2}}$

7. $A_{\text{Max.}} = 4x\sqrt{r^2 - x^2} = 4\left(\dfrac{r}{\sqrt{2}}\right)\sqrt{r^2 - \left(\dfrac{r}{\sqrt{2}}\right)^2}$

$\Rightarrow A_{\text{Max.}} = 4\left(\dfrac{r}{\sqrt{2}}\right)\sqrt{r^2 - \dfrac{r^2}{2}} = 4\left(\dfrac{r}{\sqrt{2}}\right)\sqrt{\dfrac{r^2}{2}}$

$\therefore A_{\text{Max.}} = 4\left(\dfrac{r}{\sqrt{2}}\right)\left(\dfrac{r}{\sqrt{2}}\right) = \dfrac{4r^2}{2} = 2r^2$

Answer:

8 (c) (i) $y = \sqrt{r^2 - x^2}$