CALCULUS OPTION (Q 8, PAPER 2)

LESSON NO. 2: RATIO TEST

2006

8 (c) Use the ratio test to test each of the following series for convergence. In each case, specify clearly the range of values of x for which the series converges, the range of values for which it diverges, and the value(s) of x for which the test is inconclusive.

(i) $\sum_{n=1}^{\infty} n3^n x^n$ (ii) $\sum_{n=1}^{\infty} \frac{(n+1)!n!}{(2n)!} x^n$.

SOLUTION

8 (c)

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. 2

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.
2. Find u_{n+1}.
3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

8 (c) (i)

1. $u_n = n3^n x^n$

2. $u_{n+1} = (n+1)3^{n+1}x^{n+1}$

3. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}x^{n+1}}{n3^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3x}{n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n}(1+\frac{1}{n})3x}{\cancel{n}}\right| = |3x|$

Convergent: $|3x| < 1 \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$

Divergent: $|3x| > 1 \Rightarrow x > \frac{1}{3}, x < -\frac{1}{3}$

Inconclusive: $|3x| = 1 \Rightarrow x \pm \frac{1}{3}$

8 (c) (ii)

1. $u_n = \dfrac{(n+1)!n!}{(2n)!} x^n$

2. $u_{n+1} = \dfrac{(n+2)!(n+1)!}{(2n+2)!} x^{n+1}$

3. $\lim_{n\to\infty} \left| \dfrac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \dfrac{(n+2)!(n+1)!x^{n+1}}{(2n+2)!} \times \dfrac{(2n)!}{(n+1)!n!x^n} \right|$

$= \lim_{n\to\infty} \left| \dfrac{(n+2)(n+1)x}{(2n+2)(2n+1)} \right| = \lim_{n\to\infty} \left| \dfrac{\not{n}^2(1+\frac{2}{n})(1+\frac{1}{n})x}{\not{n}^2(2+\frac{2}{n})(2+\frac{1}{n})} \right| = \left| \dfrac{x}{4} \right|$

Convergent: $\left|\frac{x}{4}\right| < 1 \Rightarrow -4 < x < 4$

Divergent: $\left|\frac{x}{4}\right| > 1 \Rightarrow x > 4, x < -4$

Inconclusive: $\left|\frac{x}{4}\right| = 1 \Rightarrow x = \pm 4$