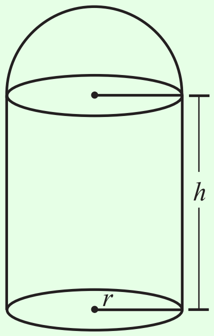
1998

8 (a) Use the ratio test to show that $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is convergent.

(b) Evaluate $\int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$.

(c) A tank with a base is made from thin uniform metal. The tank standing on level ground is in the shape of an upright circular cylinder and hemispherical top with radius of length r metres. The height of the cylinder is h metres.



(i) If the total surface area of the tank is $45\pi \text{ m}^2$, express h in terms of r.

(ii) Find the value of h and of r for which the tank has maximum volume.

SOLUTION

8 (a)

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$. 2

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = \frac{n}{2^n}$

2. $u_{n+1} = \frac{n+1}{2^{n+1}}$

3. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{2^{n+1}} \times \frac{2^n}{n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{2n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{n}(1+\frac{1}{n})}{2\cancel{n}}\right| = \frac{1}{2} < 1$

Therefore, the series is convergent.

8 (b)

Parts formula $\int u\,dv = uv - \int v\,du$ .... ❶

Steps

1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation;
   Let dv equal what is left and find v by integration.
   Note: **LIATE** helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v\,du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

List of Functions

1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1. $I = \int_0^{\frac{\pi}{2}} x^2 \cos x\,dx$

2.

$u = x^2$	$dv = \cos x\,dx$
$du = 2x\,dx$	$v = \sin x$

3. $I = (x^2)(\sin x) - \int (\sin x)2x\,dx$

$\therefore I = x^2 \sin x - 2\int x \sin x\,dx$ [Integrate by parts again.]

$u = x$	$dv = \sin x\,dx$
$du = dx$	$v = -\cos x$

$\therefore I = x^2 \sin x - 2[(x)(-\cos x) - \int (-\cos x)\,dx]$

$\therefore I = x^2 \sin x - 2[-x\cos x + \sin x]$

$\therefore I = x^2 \sin x + 2x\cos x - 2\sin x + c$

4. $I = [x^2 \sin x + 2x\cos x - 2\sin x]_0^{\frac{\pi}{2}}$

$\Rightarrow I = [((\tfrac{\pi}{2})^2 \sin(\tfrac{\pi}{2}) + 2(\tfrac{\pi}{2})\cos(\tfrac{\pi}{2}) - 2\sin(\tfrac{\pi}{2})) - (0)]$

$\Rightarrow I = [\tfrac{\pi^2}{4}(1) + \pi(0) - 2(1)]$

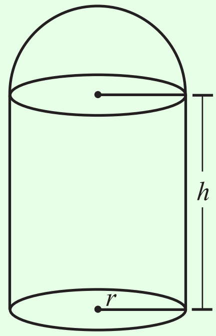
$\therefore I = \tfrac{1}{4}\pi^2 - 2$

8 (c)

STEPS
1. Identify the quantity to be maximised/minimised and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. Volume, V

2. Diagram shown.



3. $V = \pi r^2 h + \frac{2}{3}\pi r^3$

4. Surface Area, $A = 2\pi r^2 + 2\pi rh + \pi r^2 = 45\pi$

$\Rightarrow 3\pi r^2 + 2\pi rh = 45\pi$

$\Rightarrow 3r^2 + 2rh = 45$

$\Rightarrow 2rh = 45 - 3r^2$

$\Rightarrow h = \dfrac{45 - 3r^2}{2r} = \dfrac{45}{2r} - \dfrac{3r}{2}$

5. $\therefore V = \pi r^2 \left(\dfrac{45}{2r} - \dfrac{3r}{2} \right) + \frac{2}{3}\pi r^3$

$\Rightarrow V = \frac{45}{2}\pi r - \frac{3}{2}\pi r^3 + \frac{2}{3}\pi r^3$

$\Rightarrow V = \frac{45}{2}\pi r - \frac{5}{6}\pi r^3$

6. $\dfrac{dV}{dr} = 0 \Rightarrow \frac{45}{2}\pi - \frac{5}{2}\pi r^2 = 0$

$\Rightarrow 45 = 5r^2$

$\Rightarrow 9 = r^2$

$\therefore r = 3$

$\therefore h = \dfrac{45}{2(3)} - \dfrac{3(3)}{2} = \dfrac{15}{2} - \dfrac{9}{2} = 3$

ANSWERS:

8 (c) (i) $h = \frac{45}{2r} - \frac{3r}{2}$ **(ii)** $r = h = 3$ m