Calculus Option (Q 8, Paper 2)

1997

8 (a) Use the ratio test to show that

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

is convergent for all $x \in \mathbf{R}$.

(b) $f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + ...$ is the Maclaurin series.

Write the first four terms of the Maclaurin series for

$f(x) = \sqrt{1+x}$.

As your expansion converges for $-1 < x < 1$, use it to evaluate $\sqrt{10}$ correct to one place of decimals.

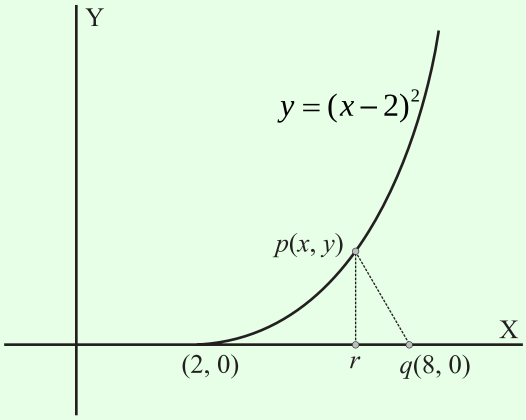
(c) $p(x, y)$ is a point on the curve $y = (x-2)^2$ in the domain $2 < x < 8$.

q is the point (8, 0) and $pr \perp rq$.

Express in terms of x, the area of the triangle prq.

What value of x maximises the area of triangle prq?

Find the maximum area of triangle prq.



Solution

8 (a)

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. 2

Steps

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = \dfrac{x^n}{n!}$

2. $u_{n+1} = \dfrac{x^{n+1}}{(n+1)!}$

3. $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)!}\times\dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{n+1}\right| = 0 < 1$

Therefore, the series is convergent for all $x \in \mathbf{R}$.

8 (b) **THE MACLAURIN FORMULA**

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \ldots + \frac{f^{(n)}(0)x^n}{n!} + \ldots$$ **3**

$f(x) = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}$

$\therefore \sqrt{1+x} = 1 + \dfrac{\frac{1}{2}x}{1!} + \dfrac{(-\frac{1}{4})x^2}{2!} + \dfrac{(\frac{3}{8})x^3}{3!}$

$\therefore \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

$\sqrt{10} = \sqrt{1+9} = \sqrt{9(1+\frac{1}{9})} = 3\sqrt{1+\frac{1}{9}}$ [x must be smaller than 1.]

$\therefore \sqrt{10} = 3(1 + \frac{1}{2}(\frac{1}{9}) - \frac{1}{8}(\frac{1}{9})^2 + \frac{1}{16}(\frac{1}{9})^3) = 3.2$ [Use your calculuator.]

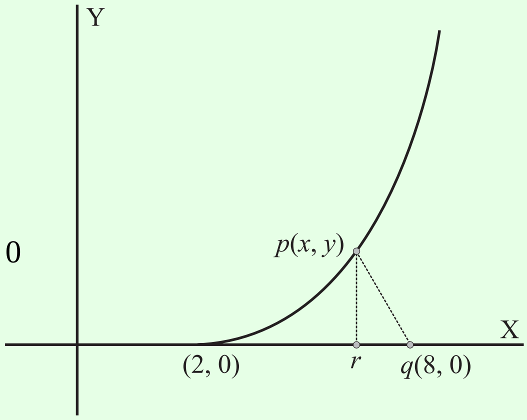
8 (c)

STEPS

1. Identify the quantity to be maximised/minimised and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. Area of triangle *prq*, *A*

2. Diagram shown.



3. $A = \frac{1}{2}bh = \frac{1}{2}(8-x)y$

4. Extra information: $y = (x-2)^2$

5. $\therefore A = \frac{1}{2}(8-x)(x-2)^2$

6. $\frac{dA}{dx} = 0 \Rightarrow \frac{1}{2}[(8-x)2(x-2) + (x-2)^2(-1)] = 0$

$\Rightarrow (x-2)\{2(8-x)-(x-2)\} = 0$

$\Rightarrow (x-2)\{16-2x-x+2\} = 0$

$\Rightarrow (x-2)\{18-3x\} = 0$

$\therefore x = 2,\ x = 6$ [Ignore x = 2 as it gives an area of zero.]

7. $A_{\text{Max.}} = \frac{1}{2}(8-6)(6-2)^2 = \frac{1}{2}(2)16 = 16$ sq. units

Answers: $\frac{1}{2}(8-x)(x-2)^2$, 6, 16