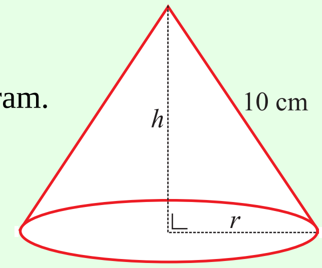
CALCULUS OPTION (Q 8, PAPER 2)

1996

8 (a) Use integration by parts to find $\int xe^{-x}dx$.

(b) The slant length of a right circular cone is 10 cm, see diagram. Find the maximum volume of the cone, in terms of π.



(c) Find $f(0),\ f'(0),\ f''(0),\ f'''(0)$ for

$f(x) = (1+x)^m$.

Hence write the first four terms and the $(r+1)$th term of the Maclaurin series for

$f(x) = (1+x)^m$.

Test the series for convergence when $m \in \mathbf{Q} \setminus \mathbf{N}$.

SOLUTION

8 (a)

PARTS FORMULA $\int u\,dv = uv - \int v\,du$ **1**

STEPS
1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation;
   Let dv equal what is left and find v by integration.
   NOTE: **LIATE** helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v\,du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

LIST of Functions
1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1. $I = \int xe^{-x}dx$

2.

$u = x$	$dv = e^{-x}dx$
$du = dx$	$v = -e^{-x}$

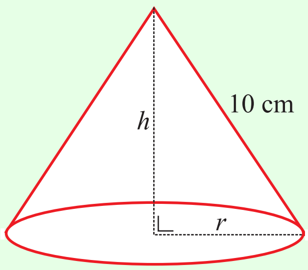
3. $I = (x)(e^{-x}) - \int(-e^{-x})dx$

$\Rightarrow I = xe^{-x} + \int e^{-x}dx$

$\therefore I = xe^{-x} - e^{-x} + c$

8 (b)

> **STEPS**
> 1. Identify the quantity to be maximised/minimised and give it a suitable symbol. **Example**: *V* for volume.
> 2. Draw a diagram (if necessary) and put in the variable(s).
> 3. Write the quantity in terms of this/these variable(s).
> 4. If there are 2 variables get rid of one in terms of the other using extra information.
> 5. Hence, write the quantity as a function of a single variable.
> 6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
> 7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.



1. Volume, V

2. Diagram shown.

3. $V = \frac{1}{3}\pi r^2 h$

4. Extra information: $h^2 + r^2 = 100 \Rightarrow r^2 = 100 - h^2$

5. $\therefore V = \frac{1}{3}\pi(100 - h^2)h$

$\Rightarrow V = \frac{100}{3}\pi h - \frac{1}{3}\pi h^3$

6. $\frac{dV}{dh} = 0 \Rightarrow \frac{100}{3}\pi - \pi h^2 = 0$

$\Rightarrow \frac{100}{3}\pi = \pi h^2$

$\therefore h = \frac{10}{\sqrt{3}}$

7. $V_{\text{Max.}} = \left(\frac{100}{3}\right)\pi\left(\frac{10}{\sqrt{3}}\right) - \left(\frac{1}{3}\right)\pi\left(\frac{10}{\sqrt{3}}\right)^3$

$\Rightarrow V_{\text{Max.}} = \left(\frac{1000}{3\sqrt{3}}\right)\pi - \left(\frac{1000}{9\sqrt{3}}\right)\pi$

$\Rightarrow V_{\text{Max.}} = \left(\frac{3000 - 1000}{9\sqrt{3}}\right)\pi$

$\therefore V_{\text{Max.}} = \frac{2000\pi}{9\sqrt{3}}$ cm^3

8 (c)

THE MACLAURIN FORMULA

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \ldots + \frac{f^{(n)}(0)x^n}{n!} + \ldots \quad \ldots\ldots \textbf{3}$$

$f(x) = (1+x)^m \Rightarrow f(0) = 1$

$f'(x) = m(1+x)^{m-1} \Rightarrow f'(0) = m$

$f''(x) = m(m-1)(1+x)^{m-2} \Rightarrow f''(0) = m(m-1)$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3} \Rightarrow f'''(0) = m(m-1)(m-2)$

$$\therefore (1+x)^m = 1 + \frac{mx}{1!} + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!}$$

$$\therefore (1+x)^m = \binom{m}{0}x^0 + \binom{m}{1}x^1 + \binom{m}{2}x^2 + \binom{m}{3}x^3$$

Steps to find general term

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series T_n to generate u_n.

   $T_n = a + (n-1)d$ **4**

2. Sometimes the signs alternate: +, −, +, −, +, −....... Multiply by $(-1)^{n-1}$ to achieve this alternation.

Powers, Binomial: 0, 1, 2, 3,..... [Arithmetic series a = 0, d = 1]

$T_r = 0 + (r-1)1 = r - 1$

Therefore, general term for $(1 + x)^m$: $u_r = \binom{m}{r-1}x^{r-1} \Rightarrow u_{r+1} = \binom{m}{r}x^r$

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. **2**

Steps

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_r = \binom{m}{r-1}x^{r-1} = \frac{m!}{(m-r+1)!(r-1)!}x^{r-1}$

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

2. $u_{r+1} = \binom{m}{r}x^r = \frac{m!}{(m-r)!r!}x^r$

3. $\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty}\left|\frac{m!x^r}{(m-r)!r!} \times \frac{(m-r+1)!(r-1)!}{m!x^{r-1}}\right| = \lim_{r\to\infty}\left|\frac{x(m-r+1)}{r}\right| = \lim_{r\to\infty}\left|\frac{x\not{r}(\frac{m}{r} - 1 + \frac{1}{r})}{\not{r}}\right| = |x|$

Therefore, the series is convergent for $|x| < 1$.