

**CALCULUS OPTION (Q 8, PAPER 2)**

**2007**

- 8 (a)  $p$  and  $q$  are real numbers such that  $p + q = 1$ .  
Find the value of  $p$  that maximizes the product  $pq$ .
- (b) (i) Derive the Maclaurin series for  $f(x) = (1+x)^m$  up to an including the term containing  $x^3$ .
- (ii) Given that the general term of the series  $f(x)$  is
- $$\frac{m(m-1)(m-2)\dots(m-r+1)}{r!} x^r,$$
- show that the series converges for  $-1 < x < 1$ .

(c) Evaluate  $\int_0^1 \tan^{-1} x \, dx$ .

**SOLUTION**

**8 (a)**

**STEPS**

1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example:**  $V$  for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. Product,  $P$
2. No diagram needed.
3.  $P = pq$
4.  $p + q = 1 \Rightarrow q = 1 - p$  [Extra information]
5.  $P = p(1 - p) = p - p^2$
6.  $\frac{dP}{dp} = 1 - 2p = 0 \Rightarrow p = \frac{1}{2}$

8 (b) (i)

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \dots + \frac{f^{(n)}(0)x^n}{n!} + \dots \quad \text{3}$$

$$f(x) = (1+x)^m \Rightarrow f(0) = 1$$

$$f'(x) = m(1+x)^{m-1} \Rightarrow f'(0) = m$$

$$f''(x) = m(m-1)(1+x)^{m-2} \Rightarrow f''(0) = m(m-1)$$

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3} \Rightarrow f'''(0) = m(m-1)(m-2)$$

$$\Rightarrow (1+x)^m = 1 + \frac{mx}{1!} + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!}$$

$$\Rightarrow (1+x)^m = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \binom{m}{3}x^3$$

8 (b) (ii)

$$\sum_{n=1}^{\infty} u_n \text{ is convergent if } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1. \text{ It is divergent if } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1. \quad \text{2}$$

STEPS

1. Read off  $u_n$  from  $\sum_{n=1}^{\infty} u_n$ .

2. Find  $u_{n+1}$ .

3. Evaluate  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right|$ . If  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$  the series is **convergent**. If

$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1$  the series is **divergent**. If  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1$  the test is **inconclusive**.

1.  $u_r = \frac{m(m-1)(m-2)\dots(m-r+1)}{r!} x^r$

2.  $u_{r+1} = \frac{m(m-1)(m-2)\dots(m-r)}{(r+1)!} x^{r+1}$

3.  $\lim_{r \rightarrow \infty} \left| \frac{u_{r+1}}{u_r} \right| = \lim_{r \rightarrow \infty} \left| \frac{m(m-1)(m-2)\dots(m-r)x^{r+1}}{(r+1)!} \times \frac{r!}{m(m-1)(m-2)\dots(m-r+1)x^r} \right|$

$$= \lim_{r \rightarrow \infty} \left| \frac{x(m-r)}{(r+1)} \right| = \lim_{r \rightarrow \infty} \left| \frac{xr(\frac{m}{r}-1)}{r(1+\frac{1}{r})} \right| = |x|$$

Series converges for  $|x| < 1 \Rightarrow -1 < x < 1$

8 (c)

$\int u dv = uv - \int v du$  ..... 1 This formula is on page 42 of the tables.

- STEPS**
1. Call the original integral  $I$  (ignore limits of integration).
  2. Let  $u$  equal the higher function in the list and find  $du$  by differentiation; Let  $dv$  equal what is left and find  $v$  by integration.  
**NOTE: LIATE** helps you to remember the order.
  3. Substitute into Parts Formula. You will now be left with  $\int v du$ . You will either be able to integrate this integral normally or you must integrate by parts again.
  4. If there are limits of integration, do them at the end.

1.  $I = \int \tan^{-1} x dx$

2.

$u = \tan^{-1} x \quad dv = 1 dx$   
 $du = \frac{1}{1+x^2} dx \quad v = x$

3.  $\therefore I = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

Find  $\int \frac{x}{1+x^2} dx$  by substitution.  $\rightarrow$

Let  $u = 1+x^2 \Rightarrow du = 2x dx \Rightarrow \frac{1}{2} du = x dx$   
 $\int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln u = \frac{1}{2} \ln(1+x^2) + c$

$\therefore I = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + c$

4.  $\therefore I = [x \tan^{-1} x - \frac{1}{2} \ln(1+x^2)]_0^1 = [(1 \tan^{-1} 1 - \frac{1}{2} \ln(1+1)) - (0 \tan^{-1} 0 - \frac{1}{2} \ln(1+0))]$   
 $= [(\frac{\pi}{4} - \frac{1}{2} \ln 2) - (0 - 0)] = \frac{\pi}{4} - \frac{1}{2} \ln 2$