CALCULUS OPTION (Q 8, PAPER 2)

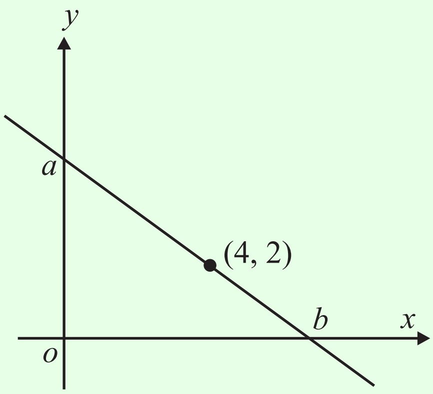
2006

8 (a) Derive the Maclaurin series for $f(x) = e^x$ up to and including the term containing x^3.

8 (b) A line passes through the point (4, 2) and has slope m, where $m < 0$. The line intersects the axes at the points a and b.

(i) Find the co-ordinates of a and b, in terms of m.

(ii) Hence, find the value of m for which the area of triangle aob is a minimum.



8 (c) Use the ratio test to test each of the following series for convergence. In each case, specify clearly the range of values of x for which the series converges, the range of values for which it diverges, and the value(s) of x for which the test is inconclusive.

(i) $\sum_{n=1}^{\infty} n3^n x^n$ (ii) $\sum_{n=1}^{\infty} \frac{(n+1)!n!}{(2n)!} x^n.$

SOLUTION

8 (a)

THE MACLAURIN FORMULA

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} ... + \frac{f^{(n)}(0)x^n}{n!} + ..$$ 3

$f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = 1$

$$e^x = \frac{1x^0}{0!} + \frac{1x^1}{1!} + \frac{1x^2}{2!} + \frac{1x^3}{3!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

8 (b) (i)

Slope $= +\frac{m}{1}$

Equation of line L: $mx - y + k = 0$

$(4, 2) \in L \Rightarrow 4m - 2 + k = 0 \Rightarrow k = 2 - 4m$

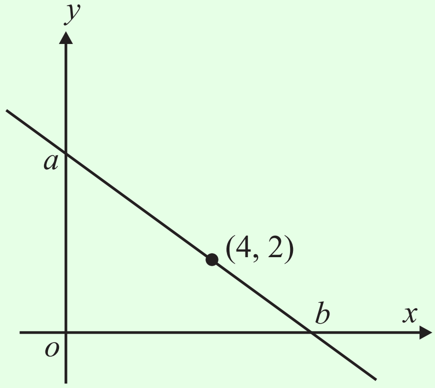
Equation of line L: $mx - y + 2 - 4m = 0$

Y intercept: Put $x = 0 \Rightarrow 2 - 4m = y$

$\therefore a(0,\ 2 - 4m)$

X intercept: Put $y = 0 \Rightarrow mx = 4m - 2 \Rightarrow x = \frac{4m-2}{m}$

$\therefore b(\frac{4m-2}{m},\ 0)$



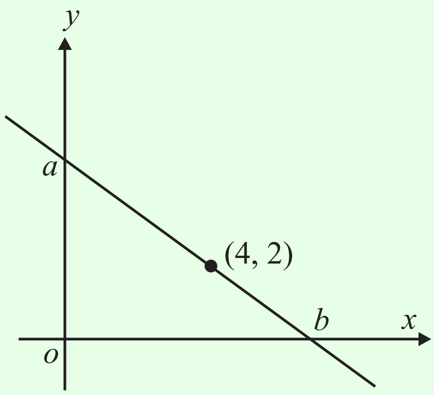
8 (b) (ii)

Steps
1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: *V* for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

1. A (Area)
2. Draw a diagram.
3. $A = \frac{1}{2}ba$
4. $a(0,\ 2-4m),\ b(\frac{4m-2}{m},\ 0)$
5. $A = \frac{1}{2}(\frac{4m-2}{m})(2-4m) = (2 - \frac{1}{m})(2-4m) = 4 - 8m - \frac{2}{m} + 4$

   $\therefore A = 8 - 8m - \frac{2}{m} = 8 - 8m - 2m^{-1}$
6. $\frac{dA}{dm} = -8 + 2m^{-2} = 0 \Rightarrow \frac{2}{m^2} = 8 \Rightarrow m^2 = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$

   $\therefore m = -\frac{1}{2},\ m < 0$



8 (c)

$\sum_{n=1}^{\infty} u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$. 2

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

8 (c) (i)

1. $u_n = n3^n x^n$

2. $u_{n+1} = (n+1)3^{n+1}x^{n+1}$

3. $$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3^{n+1}x^{n+1}}{n3^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)3x}{n}\right| = \lim_{n\to\infty}\left|\frac{\not{n}(1+\frac{1}{n})3x}{\not{n}}\right| = |3x|$$

Convergent: $|3x| < 1 \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$

Divergent: $|3x| > 1 \Rightarrow x > \frac{1}{3}, x < -\frac{1}{3}$

Inconclusive: $|3x| = 1 \Rightarrow x \pm \frac{1}{3}$

8 (c) (ii)

1. $u_n = \frac{(n+1)!n!}{(2n)!}x^n$

2. $u_{n+1} = \frac{(n+2)!(n+1)!}{(2n+2)!}x^{n+1}$

3. $$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!(n+1)!x^{n+1}}{(2n+2)!} \times \frac{(2n)!}{(n+1)!n!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)(n+1)x}{(2n+2)(2n+1)}\right| = \lim_{n\to\infty}\left|\frac{\not{n}^2(1+\frac{2}{n})(1+\frac{1}{n})x}{\not{n}^2(2+\frac{2}{n})(2+\frac{1}{n})}\right| = \left|\frac{x}{4}\right|$$

Convergent: $\left|\frac{x}{4}\right| < 1 \Rightarrow -4 < x < 4$

Divergent: $\left|\frac{x}{4}\right| > 1 \Rightarrow x > 4, x < -4$

Inconclusive: $\left|\frac{x}{4}\right| = 1 \Rightarrow x = \pm 4$