CALCULUS OPTION (Q 8, PAPER 2)

2005

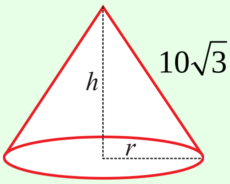
8 (a) Use integration by parts to find $\int x^2 \ln x \, dx$.

8 (b) (i) Derive the Maclaurin series for $f(x) = \ln(1+x)$ up to and including the term containing x^3.

(ii) Use those terms to find an approximation for $\ln \frac{11}{10}$.

(iii) Write down the general term of the series $f(x)$ and hence show that the series converges for $-1 < x < 1$.

8 (c) A cone has radius r cm, vertical height h cm and slant height $10\sqrt{3}$ cm. Find the value of h for which the volume is a maximum.



SOLUTION

8 (a)

PARTS FORMULA $\int u\, dv = uv - \int v\, du$ **1**

STEPS
1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation; Let dv equal what is left and find v by integration.
   NOTE: **LIATE** helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v\, du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

LIST of Functions
1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1. $I = \int x^2 \ln x \, dx$

2.

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{3}x^3$

3. $I = uv - \int v\, du = (\ln x)(\frac{1}{3}x^3) - \int \frac{1}{3}x^3(\frac{1}{x})\, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\, dx$

$\therefore I = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$

8 (b) (i) THE MACLAURIN FORMULA

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}...+\frac{f^{(n)}(0)x^n}{n!}+..$$ **3**

$$f(x)=\ln(1+x)\Rightarrow f(0)=0$$

$$f'(x)=\frac{1}{1+x}=(1+x)^{-1}\Rightarrow f'(0)=1$$

$$f''(x)=-1(1+x)^{-2}=-\frac{1}{(1+x)^2}\Rightarrow f''(0)=-1$$

$$f'''(x)=2(1+x)^{-3}=\frac{2}{(1+x)^3}\Rightarrow f'''(0)=2$$

$$\therefore \ln(1+x)=\frac{0x^0}{0!}+\frac{1x^1}{1!}-\frac{1x^2}{2!}+\frac{2x^3}{3!}=x-\frac{x^2}{2}+\frac{x^3}{3}$$

8 (b) (ii)

$$\ln\tfrac{11}{10}=\ln(1+\tfrac{1}{10})$$

Replace x by $\frac{1}{10}$ in the series formula.

$$\therefore \ln\tfrac{11}{10}=(\tfrac{1}{10})-\tfrac{1}{2}(\tfrac{1}{10})^2+\tfrac{1}{3}(\tfrac{1}{10})^3=\tfrac{1}{10}-\tfrac{1}{200}+\tfrac{1}{3000}=\tfrac{143}{1500}$$

8 (b) (iii)

STEPS TO FIND GENERAL TERM

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series T_n to generate u_n.

   $$T_n=a+(n-1)d$$ **4**

2. Sometimes the signs alternate: +, −, +, −, +, −....... Multiply by $(-1)^{n-1}$ to achieve this alternation.

Powers, Factorial: 1, 2, 3,..... [Arithmetic series a = 1, d = 1]

$$T_n=1+(n-1)1=n$$

Signs alternate.

Therefore, general term for ln (1 + x): $u_n=(-1)^{n-1}\dfrac{x^n}{n}$

$\sum_{n=1}^{\infty}u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|>1$. **2**

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ the test is **inconclusive**.

1. $u_n = (-1)^{n-1}\frac{x^n}{n}$

2. $u_{n+1} = (-1)^n \frac{x^{n+1}}{(n+1)}$

3. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)} \times \frac{n}{(-1)^{n-1} x^n}\right| = \lim_{n\to\infty}\left|(-1)^1 x \frac{n}{(n+1)}\right| = \lim_{n\to\infty}\left|(-1)^1 x \frac{n}{n(1+\frac{1}{n})}\right| = |x|$

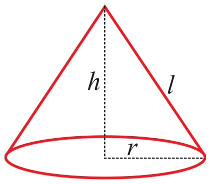
The series is convergent if $|x| < 1 \Rightarrow -1 < x < 1$

8 (c)

STEPS
1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

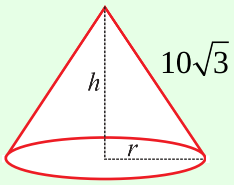
The cylinder formulae are found on page 6/7 of the tables as shown.

CONE



Curved surface area $= \pi r l$

Volume $= \frac{1}{3}\pi r^2 h$



1. V (Volume)
2. Draw a diagram.
3. $V = \frac{1}{3}\pi r^2 h$
4. $h^2 + r^2 = (10\sqrt{3})^2 \Rightarrow r^2 = 300 - h^2$ [Extra information]
5. $V = \frac{1}{3}\pi(300 - h^2)h = 100\pi h - \frac{1}{3}\pi h^3$
6. $\frac{dV}{dh} = 100\pi - \pi h^2 = 0 \Rightarrow 100 = h^2 \Rightarrow h = 10$ cm

Ans: h = 10 cm