CALCULUS OPTION (Q 8, PAPER 2)

2004

8 (a) Use integration by parts to find $\int x \sin x \, dx$.

8 (b) $f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + ..$ is the Maclaurin series.

(i) Derive the first five terms of the Maclaurin series for e^x.

(ii) Hence write down the first five terms of the Maclaurin series for e^{-x} and deduce the first three non-zero terms of the series for $\frac{e^x + e^{-x}}{2}$.

(iii) Write the general term of the series for $\frac{e^x + e^{-x}}{2}$ and use the Ratio Test to show that the series converges for all x.

8 (c) A solid cylinder has height h and radius r. The height of the cylinder, added to the circumference of its base, is equal to 3 metres.

(i) Express the volume of the cylinder in terms of r and π.

(ii) Find the maximum possible volume of the cylinder in terms of π.

SOLUTION

8 (a)

PARTS FORMULA $\int u \, dv = uv - \int v \, du$ **1**

STEPS
1. Call the original integral I (ignore limits of integration).
2. Let u equal the higher function in the list and find du by differentiation; Let dv equal what is left and find v by integration.
   NOTE: LIATE helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with $\int v \, du$. You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

LIST of Functions
1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1. $I = \int x \sin x \, dx$

2.

$u = x \qquad dv = \sin x \, dx$

$du = 1 \, dx \qquad v = -\cos x$

3. $I = uv - \int v du = -x\cos x - \int (-\cos x) \, dx = -x\cos x + \sin x + c$

8 (b) (i) THE MACLAURIN FORMULA

$$f(x)=\frac{f(0)x^0}{0!}+\frac{f'(0)x^1}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}...+\frac{f^{(n)}(0)x^n}{n!}+..$$ **3**

$f(x)=e^x \Rightarrow f(0)=1$

$f'(x)=e^x \Rightarrow f'(0)=1$

$f''(x)=e^x \Rightarrow f''(0)=1$

$f'''(x)=e^x \Rightarrow f'''(0)=1$

$f'^{v}(x)=e^x \Rightarrow f'^{v}(0)=1$

$$e^x=\frac{1x^0}{0!}+\frac{1x^1}{1!}+\frac{1x^2}{2!}+\frac{1x^3}{3!}+\frac{1x^4}{4!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$

8 (b) (ii)

Replace x by $-x$.

$$e^{-x}=1+\frac{(-x)}{1!}+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}$$

$$\frac{e^x+e^{-x}}{2}=\frac{1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}}{2}$$

$$=\frac{1+\frac{x^2}{2!}+\frac{x^4}{4!}+1+\frac{x^2}{2!}+\frac{x^4}{4!}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}$$

8 (b) (iii)

STEPS TO FIND GENERAL TERM

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series T_n to generate u_n.

$$T_n=a+(n-1)d$$ **4**

2. Sometimes the signs alternate: $+, -, +, -, +, -$....... Multiply by $(-1)^{n-1}$ to achieve this alternation.

Powers, Factorial: 0, 2, 4,..... [Arithmetic series a = 0, d = 2]

$T_n=0+(n-1)2=2n-2$

Therefore, general term for e^x: $u_n=\dfrac{x^{2n-2}}{(2n-2)!}$

$\sum_{n=1}^{\infty}u_n$ is **convergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$. It is **divergent** if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|>1$. **2**

STEPS

1. Read off u_n from $\sum_{n=1}^{\infty} u_n$.

2. Find u_{n+1}.

3. Evaluate $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$ the series is **convergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|>1$ the series is **divergent**. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=1$ the test is **inconclusive**.

1. $u_n = \dfrac{x^{2n-2}}{(2n-2)!}$

2. $u_{n+1} = \dfrac{x^{2n}}{(2n)!}$

3. $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\dfrac{x^{2n}}{(2n)!}\times\dfrac{(2n-2)!}{x^{2n-2}}\right| = \lim_{n\to\infty}\left|\dfrac{x^2}{(2n)(2n-1)}\right| = 0 < 1$

Therefore, this series is convergent for all values of x.

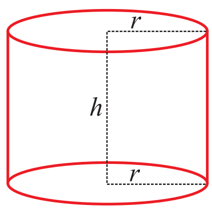
8 (c)

Use information from page 6/7 of the tables. The information you need is shown on the bottom of the page.

STEPS

1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example**: V for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

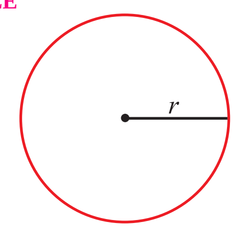
CYLINDER



Area of curved surface $= 2\pi rh$

Volume $= \pi r^2 h$

CIRCLE



Length $= 2\pi r$

Area $= \pi r^2$

1. V (Volume)
2. Draw a diagram.
3. $V = \pi r^2 h$
4. $h + 2\pi r = 3 \Rightarrow h = 3 - 2\pi r$ [Extra information]
5. $V = \pi r^2 h = \pi r^2 (3 - 2\pi r) = 3\pi r^2 - 2\pi^2 r^3$
6. $\dfrac{dV}{dr} = 6\pi r - 6\pi^2 r^2 = 0 \Rightarrow 1 - \pi r = 0 \Rightarrow r = \dfrac{1}{\pi}$
7. $V_{\text{Max}} = 3\pi(\frac{1}{\pi})^2 - 2\pi^2(\frac{1}{\pi})^3 = \frac{3}{\pi} - \frac{2}{\pi} = \frac{1}{\pi}$

8 (c) (i)

$V = 3\pi r^2 - 2\pi^2 r^3$ [Step **5**]

8 (c) (ii)

$V_{\text{Max}} = \frac{1}{\pi}$ [Step **7**]