

**CALCULUS OPTION (Q 8, PAPER 2)**

**2003**

8 (a) Use integration by parts to find  $\int xe^{-5x} dx$ .

8 (b)  $f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$  is the Maclaurin series.

(i) Derive the Maclaurin series for  $f(x) = \log_e(1+x)$  up to an including the term containing  $x^4$ .

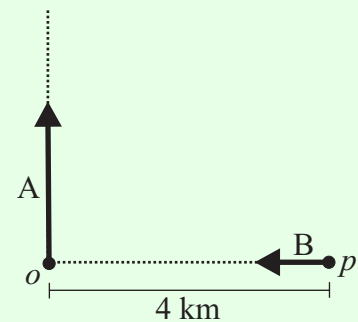
(ii) Write down the general term and use the Ratio Test to show that the series converges for  $-1 < x < 1$ .

8 (c) The point  $p$  is 4 km due east of the point  $o$ . At noon, A leaves  $o$  and travels north at a steady speed of 12 km/h. At the same time, B leaves  $p$  and travels towards  $o$  at a steady speed of 6 km/h.

(i) Write down expressions in  $x$  for the distances that A and B will each have travelled at  $x$  minutes after noon.

(ii) Find an expression in  $x$  for the distance that B will be from A at  $x$  minutes after noon.

(iii) At how many minutes after noon will B be closest to A?



**SOLUTION**

**8 (a)**

PARTS FORMULA  $\int u dv = uv - \int v du$  ..... **1**

**STEPS**

1. Call the original integral  $I$  (ignore limits of integration).
2. Let  $u$  equal the higher function in the list and find  $du$  by differentiation; Let  $dv$  equal what is left and find  $v$  by integration.  
NOTE: **LIATE** helps you to remember the order.
3. Substitute into Parts Formula. You will now be left with  $\int v du$ . You will either be able to integrate this integral normally or you must integrate by parts again.
4. If there are limits of integration, do them at the end.

**LIST of Functions**

1. **L**og
2. **I**nverse Trig
3. **A**lgebraic
4. **T**rigonometry
5. **E**xponential

1.  $I = \int xe^{-5x} dx$

2.  $u = x \quad dv = e^{-5x} dx$   
 $du = 1 dx \quad v = -\frac{1}{5}e^{-5x}$

3.  $I = x(-\frac{1}{5}e^{-5x}) - \int (-\frac{1}{5}e^{-5x})dx = -\frac{1}{5}xe^{-5x} + \frac{1}{5} \int e^{-5x} dx$

$\therefore I = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + c$

**8 (b) THE MACLAURIN FORMULA**

$$f(x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} \dots + \frac{f^{(n)}(0)x^n}{n!} + \dots \quad \text{..... } \mathbf{3}$$

**8 (b) (i)**

$$f(x) = \ln(1+x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \Rightarrow f'(0) = 1$$

$$f''(x) = -1(1+x)^{-2} = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$$

$$f^{(iv)}(x) = -6(1+x)^{-4} \Rightarrow f^{(iv)}(0) = -6$$

$$f(x) = \frac{0x^0}{0!} + \frac{1x^1}{1!} - \frac{1x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

**8 (b) (ii)**

**STEPS TO FIND GENERAL TERM**

1. The powers and coefficients of each series are in an arithmetic series. Use the formula for the general term of an arithmetic series  $T_n$  to generate  $u_n$ .

$$T_n = a + (n-1)d \quad \text{..... } \mathbf{4}$$

2. Sometimes the signs alternate: +, -, +, -, +, -, ..... Multiply by  $(-1)^{n-1}$  to achieve this alternation.

**Powers, Factorial:** 1, 2, 3,..... [Arithmetic series  $a = 1, d = 1$ ]

$$T_n = 1 + (n-1)1 = n$$

Signs alternate.

Therefore, general term for  $\sin x$ :  $u_n = (-1)^{n-1} \frac{x^n}{n}$

$$\sum_{n=1}^{\infty} u_n \text{ is } \mathbf{convergent} \text{ if } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1. \text{ It is } \mathbf{divergent} \text{ if } \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1. \quad \text{..... } \mathbf{2}$$

**STEPS**

1. Read off  $u_n$  from  $\sum_{n=1}^{\infty} u_n$ .

2. Find  $u_{n+1}$ .

3. Evaluate  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right|$ . If  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$  the series is **convergent**. If

$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1$  the series is **divergent**. If  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1$  the test is **inconclusive**.

$$1. u_n = (-1)^{n-1} \frac{x^n}{n}$$

$$2. u_{n+1} = (-1)^n \frac{x^{n+1}}{(n+1)}$$

$$3. \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n x^{n+1}}{(n+1)} \times \frac{n}{(-1)^{n-1} x^n} \right| = \lim_{n \rightarrow \infty} \left| (-1)^1 x \frac{n}{(n+1)} \right| = \lim_{n \rightarrow \infty} \left| (-1)^1 x \frac{n}{n(1 + \frac{1}{n})} \right| = |x|$$

The series is convergent if  $|x| < 1 \Rightarrow -1 < x < 1$

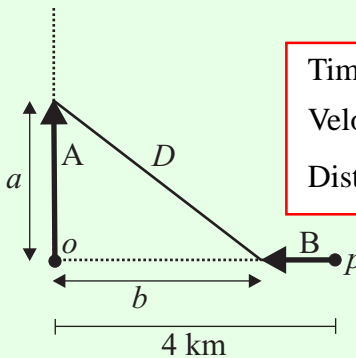
8 (c)

**STEPS**

1. Identify the quantity to be maximized/minimized and give it a suitable symbol. **Example:**  $V$  for volume.
2. Draw a diagram (if necessary) and put in the variable(s).
3. Write the quantity in terms of this/these variable(s).
4. If there are 2 variables get rid of one in terms of the other using extra information.
5. Hence, write the quantity as a function of a single variable.
6. Differentiate the quantity with respect to the variable. Set it equal to zero and solve for the variable.
7. Substitute the value of the variable back into the quantity to find the maximum/minimum value.

$$\text{Velocity} = \frac{\text{Distance}}{\text{Time}} \Rightarrow \text{Distance} = \text{Velocity} \times \text{Time}$$

Time:  $x$  minutes  
 Velocity:  $12 \text{ km/h} = \frac{1}{5} \text{ km/min}$   
 Distance:  $\frac{x}{5} \text{ km}$



Time:  $x$  minutes  
 Velocity:  $6 \text{ km/h} = \frac{1}{10} \text{ km/min}$   
 Distance:  $\frac{x}{10} \text{ km}$

1.  $D$  (Distance apart)

2. Draw a diagram.

$$3. D = \sqrt{a^2 + b^2}$$

$$4. A: a = \frac{x}{5}$$

$$B: b = 4 - \frac{x}{10}$$

$$5. D = \sqrt{\left(\frac{x}{5}\right)^2 + \left(4 - \frac{x}{10}\right)^2} = \left[\left(\frac{x}{5}\right)^2 + \left(4 - \frac{x}{10}\right)^2\right]^{\frac{1}{2}}$$

$$6. \frac{dD}{dx} = \frac{1}{2} \left[\left(\frac{x}{5}\right)^2 + \left(4 - \frac{x}{10}\right)^2\right]^{-\frac{1}{2}} \left\{ 2\left(\frac{x}{5}\right)\left(\frac{1}{5}\right) + 2\left(4 - \frac{x}{10}\right)\left(-\frac{1}{10}\right) \right\} = 0$$

$$\Rightarrow \left\{ 2\left(\frac{x}{25}\right) + 2\left(4 - \frac{x}{10}\right)\left(-\frac{1}{10}\right) \right\} = 0 \Rightarrow \frac{x}{25} - \frac{2}{5} + \frac{x}{100} = 0$$

$$\Rightarrow 4x - 40 + x = 0 \Rightarrow 5x = 40 \Rightarrow x = 8 \text{ minutes}$$

**8 (c) (i)** A:  $\frac{x}{5}$  km                      B:  $\frac{x}{10}$  km

**8 (c) (ii)**  $D = \sqrt{\left(\frac{x}{5}\right)^2 + \left(4 - \frac{x}{10}\right)^2} = \left[\left(\frac{x}{5}\right)^2 + \left(4 - \frac{x}{10}\right)^2\right]^{\frac{1}{2}}$  [Step 5]

**8 (c) (iii)** 8 minutes [Step 6]