LINE (Q 3, PAPER 2)

LESSON NO. 6: LINEAR TRANSFORMATIONS

2003

3 (a) *f* is the transformation $(x,\ y) \to (x',\ y')$ where $x' = x + y$ and $y' = x - y$. *L* is the line $4x - 2y - 1 = 0$. Find the equation of *f* (*L*), the image of *L* under *f*.

SOLUTION

3 (a)

$$\begin{array}{ll} x' = x + y & \qquad \Leftarrow x' = x + y \Rightarrow \qquad x = x + y \\ y' = x - y & \qquad \Leftarrow y' = x - y \Rightarrow \qquad -y' = -x + y \\ \hline \dfrac{x' + y'}{2} = x & \qquad\qquad\qquad\qquad \dfrac{x' - y'}{2} = y \end{array}$$

$f(L)$: $4x - 2y - 1 = 0 \Rightarrow 4\left(\dfrac{x' + y'}{2}\right) - 2\left(\dfrac{x' - y'}{2}\right) - 1 = 0$

$\Rightarrow 2(x' + y') - (x' - y') - 1 = 0 \Rightarrow 2x' + 2y' - x' + y' - 1 = 0$

$\Rightarrow x' + 3y' - 1 = 0$

2002

3 (b) *f* is the transformation $(x,\ y) \to (x',\ y')$ where $x' = 3x + y$ and $y' = x - 2y$. *S* is the square whose vertices are (0, 0), (1, 0), (1, 1) and (0, 1).

(i) Find the image of *f* of each of the four vertices of *S*.

(ii) Express *x* and *y* in terms of x' and y'.

(iii) By considering the lines $ax + by + c = 0$ and $ax + by + d = 0$, or otherwise, prove that *f* maps every pair of parallel lines. (You may assume that *f* maps every line to a line.)

(iv) Show both *S* and *f*(*S*) on a diagram.

(v) Find the area of *f*(*S*).

SOLUTION

3 (b) (i)

$(0,\ 0) \to (0,\ 0)$: $x' = 3x + y = 3(0) + (0) = 0,\ y' = x - 2y = (0) - 2(0) = 0$

$(1,\ 0) \to (3,\ 1)$: $x' = 3x + y = 3(1) + (0) = 3,\ y' = x - 2y = (1) - 2(0) = 1$

$(1,\ 1) \to (4,\ -1)$: $x' = 3x + y = 3(1) + (1) = 4,\ y' = x - 2y = (1) - 2(1) = -1$

$(0,\ 1) \to (1,\ -2)$: $x' = 3x + y = 3(0) + (1) = 1,\ y' = x - 2y = (0) - 2(1) = -2$

3 (b) (ii)

$$\begin{aligned} 2x' &= 6x + 2y \\ y' &= x - 2y \end{aligned} \quad \Leftarrow x' = 3x + y \Rightarrow \quad \begin{aligned} x' &= 3x + y \\ -3y' &= -3x + 6y \end{aligned}$$

$$\Leftarrow y' = x - 2y \Rightarrow$$

$$\frac{2x' + y'}{7} = x \qquad \frac{x' - 3y'}{7} = y$$

3 (b) (iii)

Call the two parallel lines K: $ax + by + c = 0$ and L: $ax + by + d = 0$.

$$K' : a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + c = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7c = 0$$

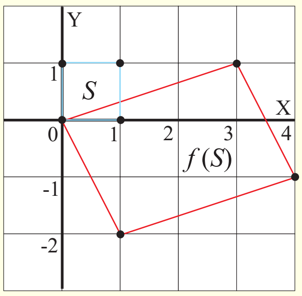
$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7c = 0$$

$$L' : a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + d = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7d = 0$$

$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7d = 0$$

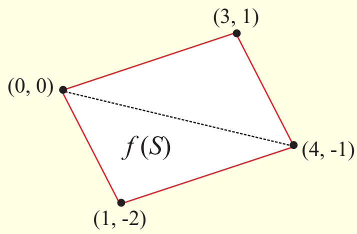
$$\therefore K \parallel L \Leftrightarrow K' \parallel L'.$$

3 (b) (iv)



3 (b) (v)

Under linear transformations:
1. Parallelograms $\rightarrow$ Parallelograms
$\square abcd \rightarrow \square a'b'c'd'$



The diagonal bisects the area of the parallelogram.

$$\therefore A = |(3)(-1) - (4)(1)| = |-3 - 4| = 7$$

$A = \frac{1}{2}|x_1y_2 - x_2y_1|$ **4**

2001

3 (b) f is the transformation $(x, y) \to (x', y')$

$$x' = -5x - 6y$$
$$y' = 4x + 3y.$$

L is the line $x - 9y = 2$.

(i) Find the equation of $f(L)$, the image of L under f.

M is a line containing the point $(1, k)$ where $k \in \mathbf{Z}$.

(ii) Given that $f(M)$ is $5x' - 2y' + 3k = 0$, find the value of k.

SOLUTION

3 (b) (i)

$$\begin{array}{ccc} x' = -5x - 6y & \Leftarrow x' = -5x - 6y \Rightarrow & 4x' = -20x - 24y \\ 2y' = 8x + 6y & \Leftarrow y' = 4x + 3y \Rightarrow & 5y' = 20x + 15y \\ \hline \dfrac{x' + 2y'}{3} = x & & -\dfrac{4x' + 5y'}{9} = y \end{array}$$

$$L: \ x - 9y - 2 = 0 \Rightarrow f(L): \left(\frac{x' + 2y'}{3}\right) - 9\left(-\frac{4x' + 5y'}{9}\right) - 2 = 0$$

$$\Rightarrow f(L): \ x' + 2y' + 12x' + 15y' - 6 = 0 \Rightarrow f(L): \ 13x' + 17y' - 6 = 0$$

3 (b) (ii)

$$f(M): \ 5x' - 2y' + 3k = 0 \Rightarrow M: \ 5(-5x - 6y) - 2(4x + 3y) + 3k = 0$$

$$\Rightarrow M: \ -25x - 30y - 8x - 6y + 3k = 0 \Rightarrow M: \ 33x + 36y - 3k = 0$$

$$\Rightarrow M: \ 11x + 12y - k = 0$$

$$(1, k) \in M \Rightarrow 11(1) + 12(k) - k = 0 \Rightarrow 11 + 12k - k = 0 \Rightarrow 11 + 11k = 0$$

$$\Rightarrow 11 = -11k \Rightarrow k = -1$$

2005

3 (c) f is the transformation $(x,\ y) \rightarrow (x',\ y'),$ where $x' = 3x - y$ and $y' = x + 2y$.

(i) Prove that f maps every pair of parallel lines to a pair of parallel lines. You may assume that f maps every line to a line.

(ii) $oabc$ is a parallelogram, where $[ob]$ is a diagonal and o is the origin. Given that $f(c) = (-1,\ 9),$ find the slope of ab.

Solution

3 (c) (i)

$$2x' = 6x - 2y \qquad \Leftarrow x' = 3x - y \Rightarrow \qquad x' = 3x - y$$
$$y' = x + 2y \qquad \Leftarrow y' = x + 2y \Rightarrow \qquad -3y' = -3x - 6y$$
$$\frac{2x' + y'}{7} = x \qquad\qquad -\frac{x' - 3y'}{7} = y$$

Call the two parallel lines K: $ax + by + c = 0$ and L: $ax + by + d = 0$.

$$K' : a\left(\frac{2x' + y'}{7}\right) + b\left(-\frac{x' - 3y'}{7}\right) + c = 0 \Rightarrow 2ax' + ay' - bx' + 3by' + 7c = 0$$

$$\Rightarrow (2a - b)x' + (a + 3b)y' + 7c = 0$$

$$L' : a\left(\frac{2x' + y'}{7}\right) + b\left(-\frac{x' - 3y'}{7}\right) + d = 0 \Rightarrow 2ax' + ay' - bx' + 3by' + 7d = 0$$

$$\Rightarrow (2a - b)x' + (a + 3b)y' + 7d = 0$$

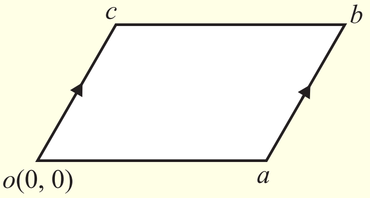
$\therefore K \parallel L \Leftrightarrow K' \parallel L'$.

3 (c) (ii)

$f(c) = (-1,\ 9) = (x',\ y')$

$$x = \frac{2x' + y'}{7} = \frac{2(-1) + (9)}{7} = 1$$

$$y = -\frac{x' - 3y'}{7} = -\frac{(-1) - 3(9)}{7} = 4$$



Therefore, the point $c(1, 4)$.

Slope of $oc = \dfrac{4 - 0}{1 - 0} = 4$

Therefore, slope of ab is 4 as $oabc$ is a parallelogram.

2004

3 (c) f is the transformation $(x,\ y) \to (x',\ y')$, where $x' = 2x - y$ and $y' = x + y$. L is the line $y = mx + c$. K is the line through the origin that is perpendicular to L.

(i) Find the equation of $f(L)$ and the equation of $f(K)$.

(ii) Find the values of m for which $f(K) \perp f(L)$. Give your answer in surd form.

SOLUTION

3 (c)

L: $mx - y + c = 0$, Slope $= +\frac{m}{1}$

K: (0, 0), Slope $= -\frac{1}{m}$

Equation of K: $x + my + k = 0 \Rightarrow (0) + m(0) + k = 0 \Rightarrow k = 0$

K: $x + my = 0$

3 (c) (i)

$$x' = 2x - y \qquad \Leftarrow x' = 2x - y \Rightarrow \qquad x' = 2x - y$$

$$y' = x + y \qquad \Leftarrow y' = x + y \Rightarrow \qquad -2y' = -2x - 2y$$

$$\frac{x' + y'}{3} = x \qquad\qquad -\frac{x' - 2y'}{3} = y$$

$f(L)$: $m\left(\frac{x'+y'}{3}\right) - \left(-\frac{x'-2y'}{3}\right) + c = 0 \Rightarrow mx' + my' + x' - 2y' + 3c = 0$

$\Rightarrow (m+1)x' + (m-2)y' + 3c = 0$

$f(K)$: $\left(\frac{x'+y'}{3}\right) + m\left(-\frac{x'-2y'}{3}\right) = 0 \Rightarrow x' + y' - mx' + 2my' = 0$

$\Rightarrow (1-m)x' + (2m+1)y' = 0$

3 (c) (ii)

> 2. **PERPENDICULAR LINES**
> Two lines are perpendicular if the product of their slopes is -1.
> $K \perp L \Leftrightarrow m_1 \times m_2 = -1$. If $m_1 = -\frac{a}{b} \Rightarrow m_2 = \frac{b}{a}$

$$\left(-\frac{1-m}{2m+1}\right)\left(-\frac{m+1}{m-2}\right) = -1 \Rightarrow (1-m)(m+1) = -1(2m+1)(m-2)$$

$$\Rightarrow m^2 - 3m - 1 = 0 \Rightarrow m = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2} = \frac{3 \pm \sqrt{9+4}}{2}$$

$$\Rightarrow m = \frac{3 \pm \sqrt{13}}{2}$$

2006

3 (c) (ii) L is the line $y = 4x$ and K is the line $x = 4y$. f is the transformation $(x, y) \to (x', y')$, where $x' = 2x - y$ and $y' = x + 3y$. Find the measure of the acute angle between $f(L)$ and $f(K)$, correct to the nearest degree.

SOLUTION

$$\begin{array}{lcl} 3x' = 6x - 3y & \Leftarrow x' = 2x - y \Rightarrow & x' = 2x - y \\ y' = x + 3y & \Leftarrow y' = x + 3y \Rightarrow & -2y' = -2x - 6y \\ \hline \frac{3x' + y'}{7} = x & & -\frac{x' - 2y'}{7} = y \end{array}$$

$$L:\ 4x - y = 0 \Rightarrow f(L):\ 4\left(\frac{3x' + y'}{7}\right) - \left(-\frac{x' - 2y'}{7}\right) = 0$$

$$\Rightarrow 12x' + 4y' + x' - 2y' = 0 \Rightarrow 13x' + 2y' = 0$$

$$K:\ x - 4y = 0 \Rightarrow f(K):\ \left(\frac{3x' + y'}{7}\right) - 4\left(-\frac{x' - 2y'}{7}\right) = 0$$

$$\Rightarrow 3x' + y' + 4x' - 8y' = 0 \Rightarrow 7x' - 7y' = 0 \Rightarrow x' - y' = 0$$

$$f(L): 13x' + 2y' = 0 \Rightarrow m_1 = -\tfrac{13}{2}$$

$$f(K): x' - y' = 0 \Rightarrow m_2 = 1$$

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) \quad \ldots\ldots\ 7$$

$$\tan\theta = \left|\frac{-\frac{13}{2} - 1}{1 + (-\frac{13}{2})(1)}\right| \Rightarrow \tan\theta = \left|\frac{-13 - 2}{2 - 13}\right| = \left|\frac{-15}{-11}\right| = \frac{15}{11}$$

$$\therefore \theta = \tan^{-1}(\tfrac{15}{11}) = 53 \cdot 7^\circ$$

Ans: 54°