LINE (Q 3, PAPER 2)

2011

3. (a) P and Q are the points (–1, 4) and (3, 7) respectively.
Find the co-ordinates of the point that divides $[PQ]$ internally in the ratio 3:1.

(b) f is the transformation $(x,\ y) \to (x',\ y')$, where $x' = x - y$ and $y' = 2x + 3y$.
l_1 is the line $2x - y - 1 = 0$.

(i) Find the equation of $f(l_1)$, the image of l_1 under f.

(ii) Prove that f maps every pair of parallel lines to a pair of parallel lines.
You may assume that f maps every line to a line.

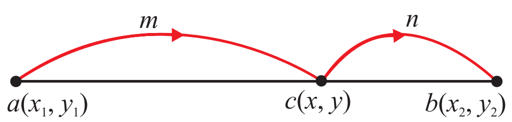
(iii) The line l_2 is parallel to the line l_1.
$f(l_2)$ intersects the x-axis at A' and the y-axis at B'.
The area of the triangle $A'OB'$ is 9 square units, where O is the origin.
Find the two possible equations of l_2.

(iv) Given that $A' = f(A)$ and $B' = f(B)$, show that $|\angle AOB| \neq |\angle A'OB'|$.

SOLUTION

3 (a)

INTERNAL DIVISION: The line segment $[ab]$ is divided internally in the ratio m:n to produce point $c(x, y)$.



$$x = \frac{mx_2 + nx_1}{m+n},\ y = \frac{my_2 + ny_1}{m+n}$$

$P(-1, 4)$ and $Q(3, 7)$
m:n = 3:1

$$\text{Point: } \left(\frac{3(3)+1(-1)}{3+1}, \frac{3(7)+1(4)}{3+1}\right) = \left(\frac{9-1}{4}, \frac{21+4}{4}\right) = \left(2, \frac{25}{4}\right)$$

3 (b)

$x' = x - y$

$y' = 2x + 3y$

$$3x' = 3x - 3y$$
$$y' = 2x + 3y$$
$$3x' + y' = 5x$$
$$x = \frac{3x' + y'}{5}$$

$$-2x' = -2x + 2y$$
$$y' = 2x + 3y$$
$$-2x' + y' = 5y$$
$$y = \frac{-2x' + y'}{5}$$

3 (b) (i)

$l_1 : 2x - y - 1 = 0$

$$f(l_1) : 2\left(\frac{3x' + y'}{5}\right) - \left(\frac{-2x' + y'}{5}\right) - 1 = 0$$
$$2(3x' + y') - (-2x' + y') - 5 = 0$$
$$6x' + 2y' + 2x' - y' - 5 = 0$$
$$8x' + y' - 5 = 0$$

3 (b) (ii)

Call the two parallel lines k: $ax + by + c = 0$ and l: $ax + by + d = 0$.

$$k' : a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + c = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7c = 0$$
$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7c = 0$$
$$l' : a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + d = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7d = 0$$
$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7d = 0$$
$$\therefore k \parallel l \Leftrightarrow k' \parallel l'.$$

3 (b) (iii)

$l_1 : 2x - y - 1 = 0$

$f(l_1) : 8x' + y' - 5 = 0$

$l_2 : 2x - y + k = 0$

$f(l_2) : 8x' + y' + 5k = 0$

> To plot a straight line you need two points. Two good points to choose are where the line cuts the axes.
> Crosses the x-axis: Put $y = 0$ (, 0)
> Crosses the y-axis: Put $x = 0$ (0,)

$A'(-\frac{5k}{8}, 0),\ B'(0, -5k)$

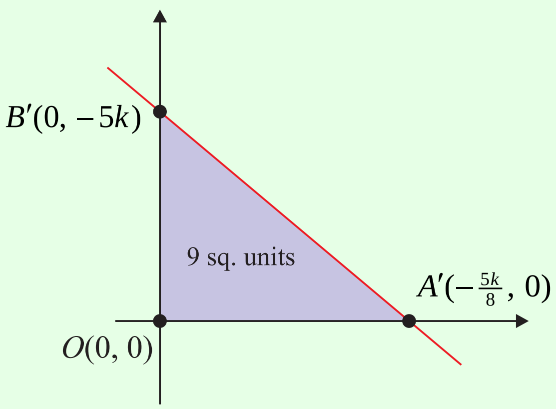
Area $= \frac{1}{2}bh$

$\therefore 9 = \frac{1}{2}(\frac{5k}{8})(5k)$

$144 = 25k^2$

$\therefore k = \pm\frac{12}{5}$

$$f(l_2) : 8x' + y' + 5(\pm\tfrac{12}{5}) = 0$$
$$8x' + y' \pm 12 = 0$$



3 (b) (iv)

$A' = f(A) = (-\tfrac{5k}{8},\ 0) = (x',\ y')$

$x = \dfrac{3x'+y'}{5} = \dfrac{3(-\tfrac{5k}{8})+(0)}{5} = \dfrac{-\tfrac{15k}{8}}{5} = -\dfrac{3k}{8}$

$y = \dfrac{-2x'+y'}{5} = \dfrac{-2(-\tfrac{5k}{8})+(0)}{5} = \dfrac{\tfrac{5k}{4}}{5} = \dfrac{k}{4}$

$\therefore A(-\tfrac{3k}{8},\ \tfrac{k}{4})$

$B' = f(B) = (0,\ -5k) = (x',\ y')$

$x = \dfrac{3x'+y'}{5} = \dfrac{3(0)+(-5k)}{5} = \dfrac{-5k}{5} = -k$

$y = \dfrac{-2x'+y'}{5} = \dfrac{-2(0)+(-5k)}{5} = \dfrac{-5k}{5} = -k$

$\therefore B(-k,\ -k)$

As is obvious from the diagram: $|\angle A'OB'| = 90^{\text{o}}$

$A(-\tfrac{3k}{8},\ \tfrac{k}{4}),\ O(0,\ 0),\ B(-k,\ -k)$

Is $AO \perp BO$?

Slope of AO: $m_1 = \dfrac{\tfrac{k}{4}-0}{-\tfrac{3k}{8}-0} = \dfrac{\tfrac{k}{4}}{-\tfrac{3k}{8}} = -\dfrac{2}{3}$

Slope of BO: $m_2 = \dfrac{0-(-k)}{0-(-k)} = \dfrac{k}{k} = 1$

Therefore, AO is not perpendicular to BO.

$|\angle AOB| \neq 90^{\text{o}} \Rightarrow |\angle AOB| \neq |\angle A'OB'|$