LINE (Q 3, PAPER 2)

2010

3 (a) The line $3x + 4y - 7 = 0$ is perpendicular to the line $ax - 6y - 1 = 0$.
Find the value of a.

(b) (i) The line $4x - 5y + k = 0$ cuts the x-axis at P and the y-axis at Q.
Write down the co-ordinates of P and Q in terms of k.

(ii) The area of the triangle OPQ is 10 square units, where O is the origin.
Find the two possible values of k.

(c) f is the transformation $(x, y) \to (x', y')$, where $x' = x + y$ and $y' = x - y$.
The line l has equation $y = mx + c$.

(i) Find the equation of $f(l)$, the image of l under f.

(ii) Find the value(s) of m for which $f(l)$ makes an angle of 45° with l.

SOLUTION

3 (a)

$3x + 4y - 7 = 0$: $m_1 = -\frac{3}{4}$

$ax - 6y - 1 = 0$: $m_2 = \frac{a}{6}$

$$m_1 \times m_2 = -1 \Rightarrow -\frac{3}{4} \times \frac{a}{6} = -1$$

$K \perp L \Leftrightarrow m_1 \times m_2 = -1.$

$$\frac{1}{4} \times \frac{a}{2} = 1$$

$$\therefore a = 8$$

3 (b) (i)

x-intercept (Put $y = 0$): $4x - 5(0) + k = 0 \Rightarrow x = -\frac{k}{4}$

$\therefore P(-\frac{k}{4}, 0)$

y-intercept (Put $x = 0$): $4(0) - 5y + k = 0 \Rightarrow y = \frac{k}{5}$

$\therefore Q(0, \frac{k}{5})$

3 (b) (ii)

$$A = \tfrac{1}{2}\left|(-\tfrac{k}{4})(\tfrac{k}{5}) - (0)(0)\right| = 10$$

$A = \frac{1}{2}|x_1 y_2 - x_2 y_1|$

$$\frac{k^2}{20} = 20$$

$$k^2 = 400$$

$$k = \pm 20$$

3 (c) (i)

$$x' = x + y \qquad x' = x + y \qquad x' = x + y$$
$$y' = x - y \quad \leftarrow \quad y' = x - y \quad \rightarrow \quad -y' = -x + y$$
$$x' + y' = 2x \qquad\qquad x' - y' = 2y$$
$$\frac{x' + y'}{2} = x \qquad\qquad \frac{x' - y'}{2} = y$$

$$l : y = mx + c$$
$$f(l) : \left(\frac{x' - y'}{2}\right) = m\left(\frac{x' + y'}{2}\right) + c$$
$$x' - y' = m(x' + y') + 2c$$
$$x' - y' = mx' + my' + 2c$$
$$x' - mx' - y' - my' - 2c = 0$$
$$-x' + mx' + y' + my' + 2c = 0$$
$$(m-1)x' + (m+1)y' + 2c = 0$$

3 (c) (ii)

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$$\theta = 45^{\circ}$$
$$l : y = mx + c \Rightarrow m_1 = m$$
$$f(l) : (m-1)x' + (m+1)y' + 2c = 0 \Rightarrow m_2 = -\frac{m-1}{m+1}$$
$$\tan 45^{\circ} = \pm\left(\frac{m - \left(-\frac{m-1}{m+1}\right)}{1 + m\left(-\frac{m-1}{m+1}\right)}\right) \Rightarrow 1 = \pm\left(\frac{m + \frac{m-1}{(m+1)}}{1 - \frac{m(m-1)}{(m+1)}}\right) \times \frac{(m+1)}{(m+1)}$$
$$\pm 1 = \frac{m(m+1) + m - 1}{(m+1) - m(m-1)}$$
$$\pm 1 = \frac{m^2 + m + m - 1}{m + 1 - m^2 + m}$$
$$\pm 1 = \frac{m^2 + 2m - 1}{-m^2 + 2m + 1}$$
$$\pm 1 = \frac{m^2 + 2m - 1}{-m^2 + 2m + 1}$$

$$1 = \frac{m^2 + 2m - 1}{-m^2 + 2m + 1}$$
$$-m^2 + 2m + 1 = m^2 + 2m - 1$$
$$-2m^2 - 2 = 0$$
$$m^2 = 1$$
$$m = \pm 1$$

$$-1 = \frac{m^2 + 2m - 1}{-m^2 + 2m + 1}$$
$$m^2 - 2m - 1 = m^2 + 2m - 1$$
$$-4m = 0$$
$$m = 0$$

$$m = -1,\ 0,\ 1$$