LINE (Q 3, PAPER 2)

2009

3 (a) Find the equation of the line through the point (1, 0) that also passes through the point of intersection of the lines $2x - y + 6 = 0$ and $10x + 3y - 2 = 0$.

(b) (i) Prove that the measure of one of the angles between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

(ii) Find the equations of the two lines that pass through the point (6, 1) and make an angle of 45° with the line $x + 2y = 0$.

(c) f is the transformation (x, y) where $(x,\ y) \to (x',\ y')$, where $x' = -x + 2y$ and $y' = 2x - y$.

(i) L is the line $ax + by + c = 0$. Prove that $f(L)$ is a line.

(ii) The line $y = mx$ is its own image under f.
Find the two possible values of m.

SOLUTION

3 (a)

Solve the two equations of the lines simultaneously to find their point of intersection.

$$\begin{aligned} 2x - y + 6 &= 0 \text{....}(\mathbf{1})(\times 3) \\ 10x + 3y - 2 &= 0 \text{...}(\mathbf{2}) \end{aligned} \quad \rightarrow \quad \begin{aligned} 6x - 3y + 18 &= 0 \\ 10x + 3y - 2 &= 0 \\ \hline 16x \quad + 16 &= 0 \Rightarrow x = -1 \end{aligned}$$

Substitute this value of x into Eqn (**1**) to find y:

$2(-1) - y + 6 = 0$

$-2 - y + 6 = 0$

$4 - y = 0$

$\therefore y = 4$

Point of intersection: (–1, 4)

Equation of line l joining (–1, 4) and (1, 0):

$$m = \frac{0-4}{1-(-1)} = \frac{-4}{2} = -2$$

$l: 2x + y + k = 0$

$(1,\ 0) \in l \Rightarrow 2(1) + (0) + k = 0$

$2 + k = 0$

$k = -2$

$l: 2x + y - 2 = 0$

3 (b) (i)

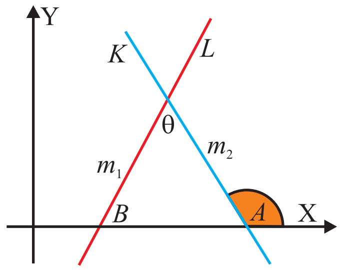
PROOF: ANGLE BETWEEN LINES FORMULA

$A = B + \theta \Rightarrow \theta = (A - B)$

$\therefore \tan\theta = \tan(A - B)$

$\Rightarrow \tan\theta = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\Rightarrow \tan\theta = \pm\left(\dfrac{m_1 - m_2}{1 + m_1 m_2}\right)$



$\pm$ is to take account of acute and obtuse cases since $\tan(180 - \theta) = -\tan\theta$.

3 (b) (ii)

Line $x + 2y = 0$: $m = -\frac{1}{2}$

$\tan\theta = \pm\left(\dfrac{m_1 - m_2}{1 + m_1 m_2}\right)$

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$\theta = 45^{\circ}$, $m_1 = -\frac{1}{2}$, $m_2 = ?$

$$\tan 45^{\circ} = \pm\left(\frac{-\frac{1}{2} - m_2}{1 + (-\frac{1}{2})m_2}\right)$$

$$\pm 1 = \left(\frac{-\frac{1}{2} - m_2}{1 - \frac{1}{2}m_2}\right) \times \frac{-2}{-2}$$

$$\pm 1 = \left(\frac{1 + 2m_2}{-2 + m_2}\right)$$

$$\pm 1 = \frac{1 + 2m_2}{-2 + m_2}$$

$$1 = \frac{1 + 2m_2}{-2 + m_2}$$

$$-2 + m_2 = 1 + 2m_2$$

$$-3 = m_2$$

$$-1 = \frac{1 + 2m_2}{-2 + m_2}$$

$$2 - m_2 = 1 + 2m_2$$

$$1 = 3m_2$$

$$\tfrac{1}{3} = m_2$$

Equation of l_1: Point (6, 1), $m = -3$

$l_1 : 3x + y + k = 0$

$(6, 1) \in l_1 \Rightarrow 3(6) + (1) + k = 0$

$19 + k = 0$

$k = -19$

$l_1 : 3x + y - 19 = 0$

Equation of l_2: Point (6, 1), $m = \frac{1}{3}$

$l_2 : x - 3y + k = 0$

$(6, 1) \in l_2 \Rightarrow (6) - 3(1) + k = 0$

$k = -3$

$l_2 : x - 3y - 3 = 0$

3 (c) (i)

$$\begin{array}{c} x' = -x + 2y \\ 2y' = 4x - 2y \\ \hline x' + 2y' = 3x \\ \dfrac{x' + 2y'}{3} = x \end{array} \quad \leftarrow \quad \begin{array}{c} x' = -x + 2y \\ y' = 2x - y \end{array} \quad \rightarrow \quad \begin{array}{c} 2x' = -2x + 4y \\ y' = 2x - y \\ \hline 2x' + y' = 3y \\ \dfrac{2x' + y'}{3} = y \end{array}$$

$L : ax + by + c = 0$

$$f(L) : a\left(\frac{x' + 2y'}{3}\right) + b\left(\frac{2x' + y'}{3}\right) + c = 0$$

$$a(x' + 2y') + b(2x' + y') + 3c = 0$$

$$ax' + 2ay' + 2bx' + by' + 3c = 0$$

$$(a + 2b)x' + (2a + b)y' + 3c = 0$$

$f(l)$ has the equation of a straight line.

3 (c) (ii)

$K : y = mx \Rightarrow \text{Slope} = m$

$$f(K) : \left(\frac{2x' + y'}{3}\right) = m\left(\frac{x' + 2y'}{3}\right)$$

$$2x' + y' = m(x' + 2y')$$

$$2x' + y' = mx' + 2my'$$

$$2x' - mx' + y' - 2my' = 0$$

$$(2 - m)x' + (1 - 2m)y' = 0$$

$$\text{Slope} = -\frac{2 - m}{1 - 2m} = \frac{m - 2}{1 - 2m}$$

As the line is an image of itself, both slopes are the same.

$$m = \frac{m - 2}{1 - 2m}$$

$$m - 2m^2 = m - 2$$

$$m^2 = 1$$

$$m = \pm 1$$