LINE (Q 3, PAPER 2)

2006

3 (a) Show that the line containing the points $(3, -6)$ and $(-7, 12)$ is perpendicular to the line $5x-9y+6=0$.

3 (b) The line K has positive slope and passes through the point $p(2, -9)$. K intersects the x-axis at q and the y-axis at r and $|pq|:|pr| = 3:1$. Find the co-ordinates of q and the co-ordinates of r.

3 (c) (i) Prove that the measure of one of the angles between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \frac{m_1 - m_2}{1+m_1m_2}.$$

(ii) L is the line $y = 4x$ and K is the line $x = 4y$. f is the transformation $(x, y) \to (x', y')$, where $x' = 2x - y$ and $y' = x + 3y$. Find the measure of the acute angle between $f(L)$ and $f(K)$, correct to the nearest degree.

SOLUTION

3 (a)

Slope between two points: $m_1 = \frac{-6-12}{3+7} = -\frac{18}{10} = -\frac{9}{5}$

$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$ 2

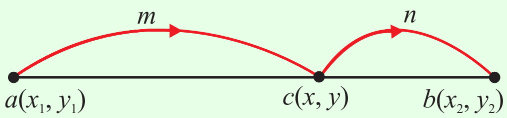
Slope of the line: $m_2 = \frac{5}{9}$

2. PERPENDICULAR LINES
Two lines are perpendicular if the product of their slopes is -1.
$K \perp L \Leftrightarrow m_1 \times m_2 = -1$. If $m_1 = -\frac{a}{b} \Rightarrow m_2 = \frac{b}{a}$

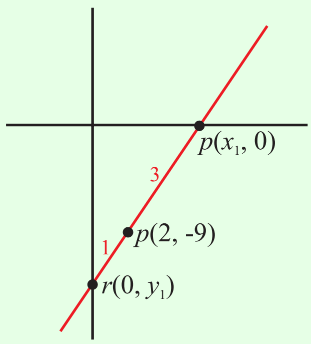
$$m_1 \times m_2 = \left(-\frac{9}{5}\right)\left(\frac{5}{9}\right) = -1$$

Therefore, the two lines are perpendicular to each other.

3 (b)



$x = \frac{mx_2 + nx_1}{m+n},\ y = \frac{my_2 + ny_1}{m+n}$ 5



p divides $[qr]$ in the ratio 3:1.

$$\therefore (2, -9) = \left(\frac{3(0)+1(x_1)}{3+1}, \frac{3(y_1)+1(0)}{3+1}\right)$$

$$\Rightarrow (2, -9) = \left(\frac{x_1}{4}, \frac{3y_1}{4}\right) \Rightarrow x_1 = 8,\ y_1 = -12$$

Ans: $p(8, 0)$, $q(0, -12)$

3 (c) (i)

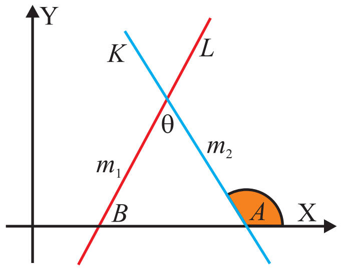
PROOF

$A = B + \theta \Rightarrow \theta = (A - B)$

$\therefore \tan\theta = \tan(A - B)$

$\Rightarrow \tan\theta = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\Rightarrow \tan\theta = \pm\left(\dfrac{m_1 - m_2}{1 + m_1 m_2}\right)$



$\pm$ is to take account of acute and obtuse cases since $\tan(180 - \theta) = -\tan\theta$.

3 (c) (ii)

$$\begin{aligned} 3x' &= 6x - 3y \\ y' &= x + 3y \end{aligned} \qquad \Leftarrow x' = 2x - y \Rightarrow \qquad \begin{aligned} x' &= 2x - y \\ -2y' &= -2x - 6y \end{aligned}$$
$$\qquad\qquad\Leftarrow y' = x + 3y \Rightarrow$$

$$\frac{3x' + y'}{7} = x \qquad\qquad -\frac{x' - 2y'}{7} = y$$

$$L:\ 4x - y = 0 \Rightarrow f(L):\ 4\left(\frac{3x' + y'}{7}\right) - \left(-\frac{x' - 2y'}{7}\right) = 0$$

$$\Rightarrow 12x' + 4y' + x' - 2y' = 0 \Rightarrow 13x' + 2y' = 0$$

$$K:\ x - 4y = 0 \Rightarrow f(K):\ \left(\frac{3x' + y'}{7}\right) - 4\left(-\frac{x' - 2y'}{7}\right) = 0$$

$$\Rightarrow 3x' + y' + 4x' - 8y' = 0 \Rightarrow 7x' - 7y' = 0 \Rightarrow x' - y' = 0$$

$$f(L):\ 13x' + 2y' = 0 \Rightarrow m_1 = -\tfrac{13}{2}$$

$$f(K):\ x' - y' = 0 \Rightarrow m_2 = 1$$

$\boxed{\tan\theta = \pm\left(\dfrac{m_1 - m_2}{1 + m_1 m_2}\right)}$ **7**

$$\tan\theta = \left|\frac{-\frac{13}{2} - 1}{1 + (-\frac{13}{2})(1)}\right| \Rightarrow \tan\theta = \left|\frac{-13 - 2}{2 - 13}\right| = \left|\frac{-15}{-11}\right| = \frac{15}{11}$$

$$\therefore \theta = \tan^{-1}(\tfrac{15}{11}) = 53 \cdot 7^\circ$$

Ans: 54°