LINE (Q 3, PAPER 2)

2005

3 (a) The line L_1: $3x-2y+7=0$ and the line L_2: $5x+y+3=0$ intersect at the point *p*. Find the equation of the line through *p* perpendicular to L_2.

3 (b) The line *K* passes through the point $(-4, 6)$ and has slope *m*, where $m > 0$.

(i) Write down the equation of *K* in terms of *m*.

(ii) Find, in terms of *m*, the co-ordinates of the points where *K* intersects the axes.

(iii) The area of the triangle formed by *K*, the *x*-axis and the *y*-axis is 54 square units. Find the possible values of *m*.

3 (c) *f* is the transformation $(x, y) \to (x', y')$, where $x' = 3x - y$ and $y' = x + 2y$.

(i) Prove that *f* maps every pair of parallel lines to a pair of parallel lines. You may assume that *f* maps every line to a line.

(ii) *oabc* is a parallelogram, where [*ob*] is a diagonal and *o* is the origin. Given that $f(c) = (-1, 9)$, find the slope of *ab*.

SOLUTION

3 (a)

You can do this using two methods:

Method 1: Find the point of intersection *p* by solving L_1 and L_2 simultaneously.

$$\begin{aligned} 3x-2y&=-7 \ldots(\mathbf{1}) \\ 10x+2y&=-6 \ldots(\mathbf{2})(\times 2) \end{aligned} \quad \rightarrow \quad \begin{aligned} 3x-2y&=-7 \\ 10x+2y&=-6 \\ \hline 13x\quad &=-13 \Rightarrow x=-1 \end{aligned}$$

Substitute this value for *x* into equation (**1**): $3(-1)-2y=-7 \Rightarrow -2y=-4 \Rightarrow y=2$

Therefore, point of intersection $p(-1, 2)$.

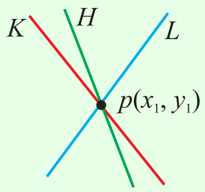
L_2: $5x+y+3=0 \Rightarrow m=-5 \Rightarrow m^{\perp}=\frac{1}{5}$

New line: $m=\frac{1}{5}, p(-1, 2)$

Equation of new line: $x-5y+k=0 \Rightarrow (-1)-5(2)+k=0 \Rightarrow -11+k=0 \Rightarrow k=11$

Ans: $x-5y+11=0$

Method 2: Use concurrent lines.



> **CONCURRENT LINE FORMULA:** Given any two concurrent lines, K and L, any other line, H, concurrent with these can be expressed as: $H = \mu K + \lambda L$ where $\mu, \lambda \in \mathbf{R}$.

Equation of new line: $\mu L_1 + \lambda L_2 = 0 \Rightarrow \mu(3x - 2y + 7) + \lambda(5x + y + 3) = 0$

$$\Rightarrow (3\mu + 5\lambda)x + (-2\mu + \lambda)y + 7\mu + 3\lambda = 0 \Rightarrow m = \frac{3\mu + 5\lambda}{2\mu - \lambda}$$

This line is perpendicular to L_2:

$$\Rightarrow \frac{3\mu + 5\lambda}{2\mu - \lambda} = \frac{1}{5} \Rightarrow 15\mu + 25\lambda = 2\mu - \lambda \Rightarrow 13\mu = -26\lambda \Rightarrow \mu = -2\lambda$$

Substitute this value of μ into the equation of the new line:

$$\Rightarrow (-6\lambda + 5\lambda)x + (4\lambda + \lambda)y - 14\lambda + 3\lambda = 0 \Rightarrow -\lambda x + 5\lambda y - 11\lambda = 0$$

$$\Rightarrow x - 5y + 11 = 0$$

3 (b) (i)

Slope $= +\frac{m}{1}$, Point $(-4, 6)$

Equation of K: $mx - y + k = 0 \Rightarrow m(-4) - (6) + k = 0 \Rightarrow k = 4m + 6$

K: $mx - y + 4m + 6 = 0$

3 (b) (ii)

> To plot a straight line you need 2 points. Two good points to choose are where the line cuts the axes.
> 1. Crosses the X-axis: Put $y = 0$ (, 0)
> 2. Crosses the Y-axis: Put $x = 0$ (0,)

X intercept: $mx - (0) + 4m + 6 = 0 \Rightarrow mx = -4m - 6 \Rightarrow x = \dfrac{-4m - 6}{m}$

Y intercept: $m(0) - y + 4m + 6 = 0 \Rightarrow y = 4m + 6$

Intercepts: $\left(\dfrac{-4m-6}{m}, 0\right), (0, 4m + 6)$

3 (b) (iii)

$$A = \tfrac{1}{2}\left|\left(\frac{-4m-6}{m}\right)(4m+6) - (0)(0)\right| = 54$$

$A = \frac{1}{2}|x_1y_2 - x_2y_1|$ **4**

$$\Rightarrow \left|-\left(\frac{4m+6}{m}\right)(2m+3)\right| = 54 \Rightarrow (4m+6)(2m+3) = 54m$$

$$\Rightarrow (2m+3)(2m+3) = 27m \Rightarrow 4m^2 + 12m + 9 = 27m$$

$$\Rightarrow 4m^2 - 15m + 9 = 0 \Rightarrow (4m-3)(m-3) = 0 \Rightarrow m = \tfrac{3}{4}, 3$$

3 (c) (i)

$$\begin{array}{lll} 2x' = 6x - 2y & \Leftarrow x' = 3x - y \Rightarrow & x' = 3x - y \\ y' = x + 2y & \Leftarrow y' = x + 2y \Rightarrow & -3y' = -3x - 6y \\ \hline \frac{2x' + y'}{7} = x & & -\frac{x' - 3y'}{7} = y \end{array}$$

Call the two parallel lines K: $ax + by + c = 0$ and L: $ax + by + d = 0$.

$$K' : a\left(\frac{2x' + y'}{7}\right) + b\left(-\frac{x' - 3y'}{7}\right) + c = 0 \Rightarrow 2ax' + ay' - bx' + 3by' + 7c = 0$$

$$\Rightarrow (2a - b)x' + (a + 3b)y' + 7c = 0$$

$$L' : a\left(\frac{2x' + y'}{7}\right) + b\left(-\frac{x' - 3y'}{7}\right) + d = 0 \Rightarrow 2ax' + ay' - bx' + 3by' + 7d = 0$$

$$\Rightarrow (2a - b)x' + (a + 3b)y' + 7d = 0$$

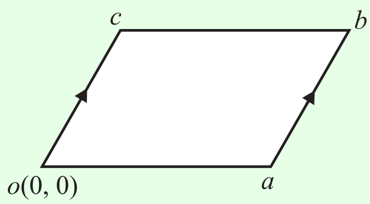
$\therefore K \parallel L \Leftrightarrow K' \parallel L'$.

3 (c) (ii)

$$f(c) = (-1, 9) = (x', y')$$

$$x = \frac{2x' + y'}{7} = \frac{2(-1) + (9)}{7} = 1$$

$$y = -\frac{x' - 3y'}{7} = -\frac{(-1) - 3(9)}{7} = 4$$



Therefore, the point $c(1, 4)$.

Slope of $oc = \frac{4 - 0}{1 - 0} = 4$

Therefore, slope of ab is 4 as $oabc$ is a parallelogram.