LINE (Q 3, PAPER 2)

2002

3 (a) $a(-1, 4)$ and $b(5, -4)$ are two points. Find the equation of the perpendicular bisector of $[ab]$.

3 (b) f is the transformation $(x, y) \to (x', y')$ where $x' = 3x + y$ and $y' = x - 2y$. S is the square whose vertices are (0, 0), (1, 0), (1, 1) and (0, 1).

(i) Find the image of f of each of the four vertices of S.

(ii) Express x and y in terms of x' and y'.

(iii) By considering the lines $ax + by + c = 0$ and $ax + by + d = 0$, or otherwise, prove that f maps every pair of parallel lines. (You may assume that f maps every line to a line.)

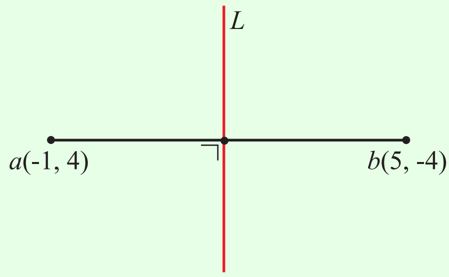
(iv) Show both S and $f(S)$ on a diagram.

(v) Find the area of $f(S)$.

SOLUTION

3 (a)

The perpendicular bisector L passes through the midpoint of $[ab]$ and has a perpendicular slope to ab.



Midpoint of $[ab]$: $\left(\frac{-1+5}{2}, \frac{4-4}{2}\right) = (2, 0)$

$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$ 2

2. PERPENDICULAR LINES
Two lines are perpendicular if the product of their slopes is -1.
$K \perp L \Leftrightarrow m_1 \times m_2 = -1$. If $m_1 = -\frac{a}{b} \Rightarrow m_2 = \frac{b}{a}$

Slope of ab: $m_1 = \frac{4+4}{-1-5} = -\frac{8}{6} = -\frac{4}{3}$

Therefore, slope of L: $m_2 = \frac{3}{4}$

Equation of a line: $y - y_1 = m(x - x_1)$ 3

Equation of L: $3x - 4y + k = 0 \Rightarrow 3(2) - 4(0) + k = 0 \Rightarrow k = -6$

L: $3x - 4y - 6 = 0$

3 (b) (i)

$(0,\ 0) \to (0,\ 0)$: $x' = 3x + y = 3(0) + (0) = 0,\ y' = x - 2y = (0) - 2(0) = 0$

$(1,\ 0) \to (3,\ 1)$: $x' = 3x + y = 3(1) + (0) = 3,\ y' = x - 2y = (1) - 2(0) = 1$

$(1,\ 1) \to (4,\ -1)$: $x' = 3x + y = 3(1) + (1) = 4,\ y' = x - 2y = (1) - 2(1) = -1$

$(0,\ 1) \to (1,\ -2)$: $x' = 3x + y = 3(0) + (1) = 1,\ y' = x - 2y = (0) - 2(1) = -2$

3 (b) (ii)

$$\begin{array}{lcl} 2x' = 6x + 2y & \Leftarrow x' = 3x + y \Rightarrow & x' = 3x + y \\ y' \ \ = x - 2y & \Leftarrow y' = x - 2y \Rightarrow & -3y' = -3x + 6y \\ \hline \dfrac{2x' + y'}{7} = x & & \dfrac{x' - 3y'}{7} = y \end{array}$$

3 (b) (iii)

Call the two parallel lines K: $ax + by + c = 0$ and L: $ax + by + d = 0$.

$$K' : a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + c = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7c = 0$$

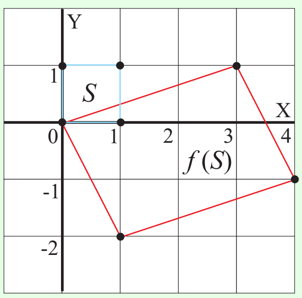
$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7c = 0$$

$$L' : a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + d = 0 \Rightarrow 2ax' + ay' + bx' - 3by' + 7d = 0$$

$$\Rightarrow (2a + b)x' + (a - 3b)y' + 7d = 0$$

$\therefore K \parallel L \Leftrightarrow K' \parallel L'$.

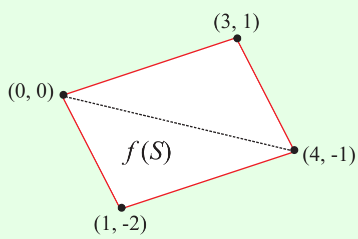
3 (b) (iv)



3 (b) (v)

Under linear transformations:
1. Parallelograms $\rightarrow$ Parallelograms

$\square abcd \rightarrow \square a'b'c'd'$



The diagonal bisects the area of the parallelogram.

$\therefore A = |(3)(-1) - (4)(1)| = |-3 - 4| = 7$

$A = \frac{1}{2}|x_1y_2 - x_2y_1|$ **4**