Integration (Q 8, Paper 1)

Lesson No. 7: Applications of Integration I: Area

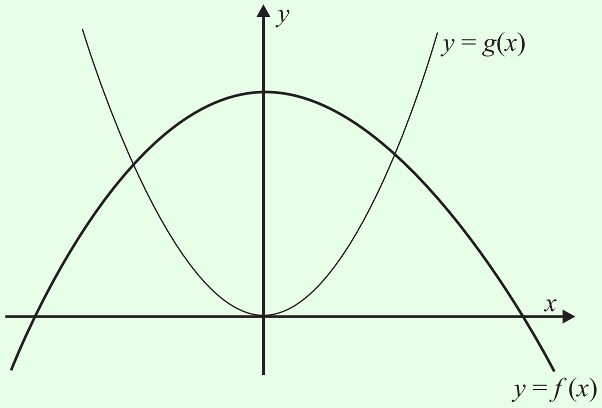
2006

8 (c) The diagram shows the graphs of the curves $y = f(x)$ and $y = g(x)$, where

$f(x) = 12 - 3x^2$ and $g(x) = 9x^2$.

(i) Calculate the area of the region enclosed by the curve $y = f(x)$ and the x-axis.

(ii) Show that the region enclosed by the curves $y = f(x)$ and $y = g(x)$ has half that area.

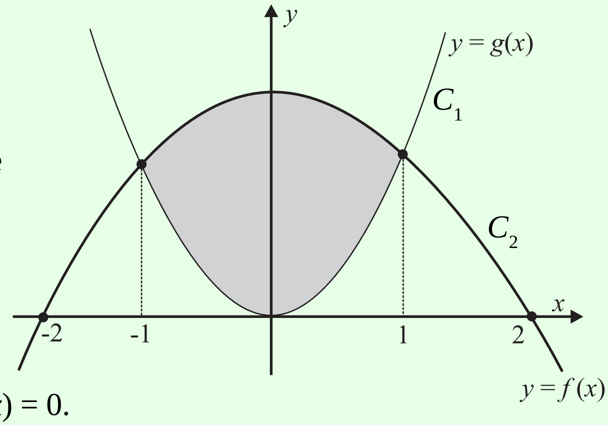


Solution

8 (c) (i)

To find the area between the curve $y = f(x)$ and the x-axis, you need to find where the curve intersects the x-axis and then integrate the curve between these limits.

$$A = \int_a^b y\,dx \quad \text{.......} \quad \textbf{8}$$



Intersection of the curve with x-axis: Set $f(x) = 0$.

$12 - 3x^2 = 0 \Rightarrow 12 = 3x^2 \Rightarrow x = \pm 2$

Area under C_1:

$$\int_{-2}^{2} (12 - 3x^2)\,dx = [12x - x^3]_{-2}^{2} = [(12 \times 2 - 2^3) - (12(-2) - (-2)^3)]$$

$$= [(24 - 8) - (-24 + 8)] = [(16) - (-16)] = 32$$

8 (c) (ii)

1. $f(x) = g(x) \Rightarrow 12 - 3x^2 = 9x^2$

$\Rightarrow 12 = 12x^2 \Rightarrow x = \pm 1$

2. A = Area under C_1 − Area under C_2

$$\Rightarrow A = \int_{-1}^{1} (12 - 3x^2)\,dx - \int_{-1}^{1} 9x^2\,dx$$

$$= [12x - x^3]_{-1}^{1} - [3x^3]_{-1}^{1} = [9 - (-11)] - [3 - (-3)] = 20 - 6 = 16$$

Two curve problems

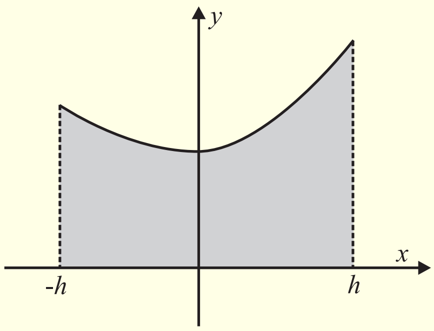
1. Find the points of intersection of the curves by solving simultaneously.
2. The area A between the curve C_1 and the curve C_2 is given by:
   A = Area under C_1 − Area under C_2

2004

8 (c) The graph of the function $f(x) = ax^2 + bx + c$ from $x = -h$ to $x = h$ is shown in the diagram.

(i) Show that the area of the shaded region is $\frac{h}{3}[2ah^2 + 6c]$.

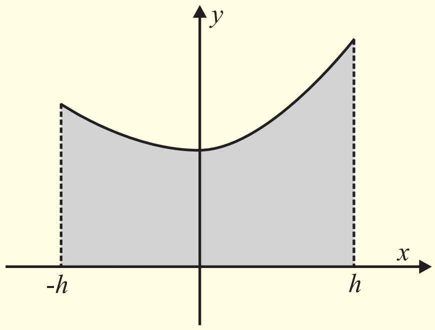
(ii) Given that $f(-h) = y_1$, $f(0) = y_2$ and $f(h) = y_3$, express the area of the shaded region in terms of y_1, y_2, y_3 and h.



Solution

8 (c) (i)

$$A = \int_a^b y\, dx \quad \text{.......} \; 8$$



$$A = \int_{-h}^{h} (ax^2 + bx + c)\, dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$

$$= \left[\left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch\right) - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - ch\right)\right]$$

$$= \left[\frac{ah^3}{3} + \frac{bh^2}{2} + ch + \frac{ah^3}{3} - \frac{bh^2}{2} + ch\right] = \left[\frac{2ah^3}{3} + 2ch\right] = \frac{h}{3}[2ah^2 + 6c]$$

8 (c) (ii)

$$f(-h) = ah^2 - bh + c = y_1$$

$$f(h) = ah^2 + bh + c = y_3$$

$$\Rightarrow y_1 + y_3 = 2ah^2 + 2c$$

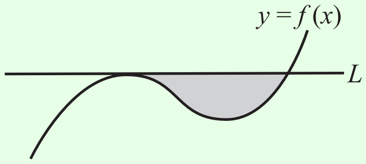
$$f(0) = a(0)^2 + b(0) + c = y_2 \Rightarrow c = y_2$$

$$A = \frac{h}{3}[2ah^2 + 6c] = \frac{h}{3}[2ah^2 + 2c + 4c] = \frac{h}{3}[y_1 + y_3 + 4y_2]$$

2002

8 (c) Let $f(x) = x^3 - 3x^2 + 5$.

L is the tangent to the curve $y = f(x)$ at its local maximum point.
Find the area enclosed between L and the curve.



SOLUTION

Firstly, find the local maximum point.

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ **11**

To find out if the turning point (TP) is a local maximum or local minimum:

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{TP} < 0$

Local Mininimum: $\left(\frac{d^2y}{dx^2}\right)_{TP} > 0$ **12**

$y = x^3 - 3x^2 + 5 \Rightarrow \frac{dy}{dx} = 3x^2 - 6x$

Put $\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0,\ 2$

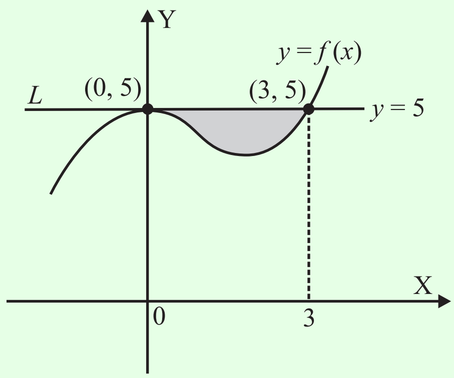
At $x = 0 \Rightarrow (0)^3 - 3(0)^2 + 5 = 5 \Rightarrow (0,\ 5)$ is a turning point.

$x = 2 \Rightarrow (2)^3 - 3(2)^2 + 5 = 1 \Rightarrow (2,\ 1)$ is a turning point.

Which turning point is the local maximum point?

$\frac{d^2y}{dx^2} = 6x - 6 \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=0} = 6(0) - 6 = -6 < 0 \Rightarrow$ (0, 5) is the local maximum point.

Now work on the diagram putting the axes in the right place.
As you can see, the line L has equation $y = 5$.
To find the other point of intersection between the line ($y = 5$) and the curve ($y = x^3 - 3x^2 + 5$), equate these equations and solve for x.



$\Rightarrow 5 = x^3 - 3x^2 + 5 \Rightarrow x^3 - 3x^2 = 0 \Rightarrow x^2(x-3) = 0$

$\Rightarrow x = 0,\ 3$

The points of intersection are (0, 5) and (3, 5).

Area of shaded part = Area under L – Area under the curve

$\therefore A = \int_0^3 5\,dx - \int_0^3 (x^3 - 3x^2 + 5)\,dx$

$= [5x]_0^3 - [\tfrac{1}{4}x^4 - x^3 + 5x]_0^3 = [15 - 0] - [(\tfrac{1}{4}(3)^4 - (3)^3 + 5(3)) - (0)]$

$= 15 - \tfrac{81}{4} + 27 - 15 = \tfrac{27}{4}$

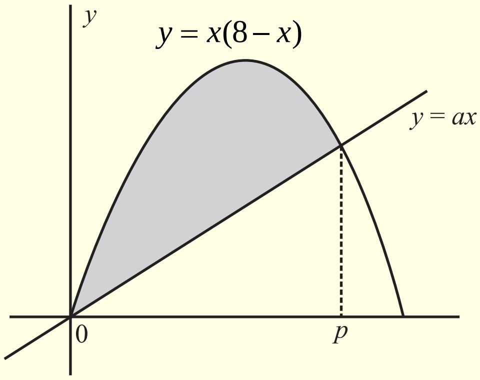
2001

8 (c) a is a real number such that $0 < a < 8$.

The line $y = ax$ intersects the curve $y = x(8-x)$ at $x = 0$ and at $x = p$.

(i) Show that $p = 8 - a$.

(ii) Show that the area between the curve and the line is $\frac{p^3}{6}$ square units.



SOLUTION

8 (c) (i)

Equate the line ($y = ax$) and the curve ($y = x(8-x)$) to find out where they intersect. You are also told they intersect at $x = 0$ and $x = p$.

$\Rightarrow ax = x(8-x) \Rightarrow ax = 8x - x^2$

At $x = p$: $\Rightarrow ap = 8p - p^2 \Rightarrow a = 8 - p \Rightarrow p = 8 - a$

8 (c) (ii)

Shaded area = Area under the curve − Area under the line

$$\therefore A = \int_0^p (8x - x^2)dx - \int_0^p ax\,dx$$

$$= [4x^2 - \tfrac{1}{3}x^3]_0^p - [\tfrac{1}{2}ax^2]_0^p = [(4p^2 - \tfrac{1}{3}p^3) - (0)] - [(\tfrac{1}{2}ap^2) - (0)]$$

$$= 4p^2 - \tfrac{1}{3}p^3 - \tfrac{1}{2}ap^2 = 4p^2 - \tfrac{1}{3}p^3 - \tfrac{1}{2}(8-p)p^2$$

$$= 4p^2 - \tfrac{1}{3}p^3 - 4p^2 + \tfrac{1}{2}p^3 = \frac{p^3}{6}$$