

INTEGRATION (Q 8, PAPER 1)

2011

8. (a) Find $\int (x^3 + \sqrt{x}) dx$.

(b) (i) Evaluate $\int_0^2 \frac{x+1}{x^2+2x+2} dx$.

(ii) Evaluate $\int_0^2 \frac{x^2+2x+2}{x+1} dx$.

(c) Use integration methods to establish the formula $A = \pi r^2$ for the area of a disc of radius r .

SOLUTION

8 (a)

$$\begin{aligned} I &= \int (x^3 + \sqrt{x}) dx = \int (x^3 + x^{\frac{1}{2}}) dx \\ &= \frac{x^4}{4} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{1}{4}x^4 + \frac{2}{3}x^{\frac{3}{2}} + c \end{aligned}$$

$$\int x^p dx = \frac{x^{p+1}}{p+1} + c \text{ for } p \in \mathbf{R} \text{ except } p \neq -1.$$

Remember it as:

Add one to the power of x and divide by the new power.

8 (b) (i)

STEPS

1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x .
3. Change the limits from x to u .
4. Make the substitution.
5. Evaluate the integral I .

$$\begin{aligned} I &= \int_0^2 \frac{x+1}{x^2+2x+2} dx \\ &= \frac{1}{2} \int_2^{10} \frac{du}{u} \\ &= \frac{1}{2} [\ln u]_2^{10} \\ &= \frac{1}{2} (\ln 10 - \ln 2) \\ &= \frac{1}{2} \ln\left(\frac{10}{2}\right) \\ &= \frac{1}{2} \ln 5 \end{aligned}$$

Make a substitution:

$$u = x^2 + 2x + 2$$

$$du = (2x + 2) dx$$

$$du = 2(x + 1) dx$$

$$\frac{1}{2} du = (x + 1) dx$$

Change limits:

$$x = 2 \Rightarrow u = (2)^2 + 2(2) + 2 = 10$$

$$x = 0 \Rightarrow u = (0)^2 + 2(0) + 2 = 2$$

8 (b) (i)

$$\begin{aligned} I &= \int_0^2 \frac{x^2 + 2x + 2}{x + 1} dx \\ &= \int_1^3 \frac{(u-1)^2 + 2(u-1) + 2}{u} du \\ &= \int_1^3 \frac{u^2 - 2u + 1 + 2u - 2 + 2}{u} du \\ &= \int_1^3 \frac{u^2 + 1}{u} du \\ &= \int_1^3 \left(u + \frac{1}{u}\right) du \\ &= \left[\frac{1}{2}u^2 + \ln u\right]_1^3 \\ &= \left[\frac{1}{2}(9) + \ln 3\right] - \left[\frac{1}{2}(1) + \ln 1\right] \\ &= \left[\frac{9}{2} + \ln 3 - \frac{1}{2} - \ln 1\right] \\ &= 4 + \ln 3 \end{aligned}$$

Make a substitution:

$$u = x + 1 \Rightarrow x = (u - 1)$$

$$du = dx$$

Change limits:

$$x = 2 \Rightarrow u = (2) + 1 = 3$$

$$x = 0 \Rightarrow u = (0) + 1 = 1$$

8 (c)

$$A = 4 \int_0^r y dx = 4 \int_0^r \sqrt{r^2 - x^2} dx$$

$$\text{Let } x = r \sin u$$

$$dx = r \cos u du$$

$$A = 4 \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 u} r \cos u du$$

$$= 4r \int_0^{\frac{\pi}{2}} \sqrt{r^2(1 - \sin^2 u)} \cos u du \quad [\text{Use } \cos^2 u + \sin^2 u = 1]$$

$$= 4r \int_0^{\frac{\pi}{2}} \sqrt{r^2 \cos^2 u} \cos u du$$

$$= 4r \int_0^{\frac{\pi}{2}} r \cos u \cos u du$$

$$= 4r^2 \int_0^{\frac{\pi}{2}} \cos^2 u du \quad \boxed{\cos^2 A = \frac{1}{2}(1 + \cos 2A)}$$

$$= 4r^2 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2u) du$$

$$= 2r^2 \int_0^{\frac{\pi}{2}} (1 + \cos 2u) du$$

$$= 2r^2 \left[u + \frac{1}{2} \sin 2u \right]_0^{\frac{\pi}{2}}$$

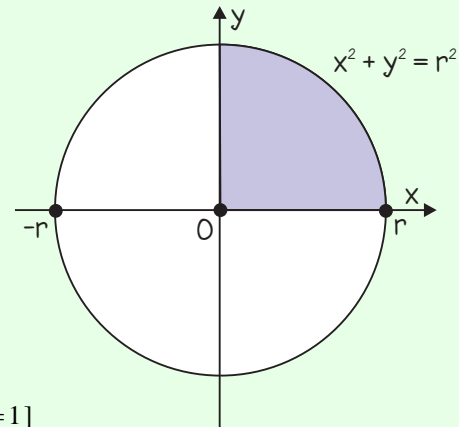
$$= 2r^2 \left[\left(\frac{\pi}{2} + \frac{1}{2} \sin 2\left(\frac{\pi}{2}\right) \right) - \left(0 + \frac{1}{2} \sin 2(0) \right) \right]$$

$$= 2r^2 \left[\frac{\pi}{2} + \frac{1}{2} \sin \pi - 0 - \frac{1}{2} \sin 0 \right]$$

$$= 2r^2 \left[\frac{\pi}{2} + \frac{1}{2}(0) - 0 - \frac{1}{2}(0) \right]$$

$$= 2r^2 \left[\frac{\pi}{2} \right]$$

$$= \pi r^2$$



Change limits:

$$x = r: r = r \sin u$$

$$1 = \sin u$$

$$\therefore u = \sin^{-1} 1 = \frac{\pi}{2}$$

$$x = 0: 0 = r \sin u$$

$$0 = \sin u$$

$$\therefore u = \sin^{-1} 0 = 0$$