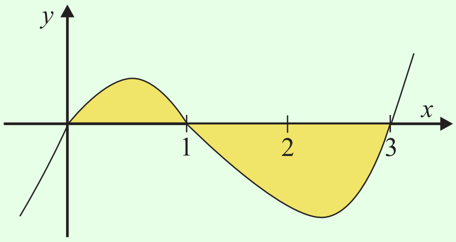
INTEGRATION (Q 8, PAPER 1)

2010

8 (a) Find $\int (\sin 2x + e^{4x})dx$.

(b) The curve $y = 12x^3 - 48x^2 + 36x$ crosses the x-axis at $x = 0$, $x = 1$ and $x = 3$, as shown.
Calculate the total area of the shaded regions enclosed by the curve and the x-axis.



(c) (i) Find, in terms of a and b,

$$I = \int_a^b \frac{\cos x}{1+\sin x} dx$$

(ii) Find in terms of a and b,

$$J = \int_a^b \frac{\sin x}{1+\cos x} dx.$$

(iii) Show that if $a + b = \frac{\pi}{2}$, then $I = J$.

SOLUTION

8 (a)

$$\int (\sin 2x + e^{4x})\, dx$$

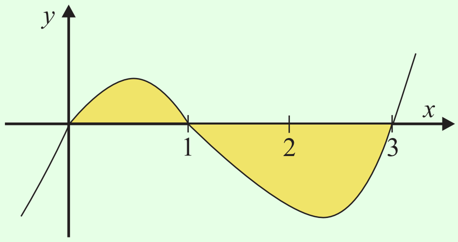
$$= \frac{-\cos 2x}{2} + \frac{e^{4x}}{4} + c$$

$$= -\tfrac{1}{2}\cos 2x + \tfrac{1}{4}e^{4x} + c$$

$$\int \sin(ax+b)dx = -\tfrac{1}{a}\cos(ax+b) + c$$

$$\int e^{ax+b} dx = \tfrac{1}{a}e^{ax+b} + c$$

8 (b)



$$A = |A_1| + |A_2|$$

$$= \left|\int_0^1 (12x^3 - 48x^2 + 36x)\,dx\right| + \left|\int_1^3 (12x^3 - 48x^2 + 36x)\,dx\right|$$

$$= \left|\left[\frac{12x^4}{4} - \frac{48x^3}{3} + \frac{36x^2}{2}\right]_0^1\right| + \left|\left[\frac{12x^4}{4} - \frac{48x^3}{3} + \frac{36x^2}{2}\right]_1^3\right|$$

$$= \left|\left[3x^4 - 16x^3 + 18x^2\right]_0^1\right| + \left|\left[3x^4 - 16x^3 + 18x^2\right]_1^3\right|$$

$$= \left|(3(1)^4 - 16(1)^3 + 18(1)^2) - (3(0)^4 - 16(0)^3 + 18(0)^2)\right|$$

$$+ \left|(3(3)^4 - 16(3)^3 + 18(3)^2) - (3(1)^4 - 16(1)^3 + 18(1)^2)\right|$$

$$= |(3 - 16 + 18) - (0 - 0 + 0)| + |(243 - 432 + 162) - (3 - 16 + 18)|$$

$$= |(5) - (0)| + |(-27) - (5)|$$

$$= |5| + |-27 - 5|$$

$$= |5| + |-32|$$

$$= 5 + 32$$

$$= 37$$

8 (c) (i)

$$I = \int_a^b \frac{\cos x}{1 + \sin x}\,dx$$

Make a substitution:
$u = 1 + \sin x \Rightarrow du = \cos x\,dx$

Change limits:
$x = b \Rightarrow u = 1 + \sin b$
$x = a \Rightarrow u = 1 + \sin a$

$$\therefore I = \int_{1+\sin a}^{1+\sin b} \frac{du}{u} = [\ln u]_{1+\sin a}^{1+\sin b} = [\ln(1 + \sin b) - \ln(1 + \sin a)] = \ln\left[\frac{1 + \sin b}{1 + \sin a}\right]$$

8 (c) (ii)

$$J = \int_a^b \frac{\sin x}{1 + \cos x}\,dx$$

Make a substitution:
$u = 1 + \cos x \Rightarrow du = -\sin x\,dx$

$$\therefore J = -\int_{1+\cos a}^{1+\cos b} \frac{du}{u} = -[\ln u]_{1+\cos a}^{1+\cos b}$$

$$= -[\ln(1 + \cos b) - \ln(1 + \cos a)]$$

$$= [\ln(1 + \cos a) - \ln(1 + \cos b)]$$

$$= \ln\left[\frac{1 + \cos a}{1 + \cos b}\right]$$

Change limits:
$x = b \Rightarrow u = 1 + \cos b$
$x = a \Rightarrow u = 1 + \cos a$

8 (c) (iii)

$$I = \ln\left[\frac{1+\sin b}{1+\sin a}\right]$$

$$a+b=\tfrac{\pi}{2} \Rightarrow a=\tfrac{\pi}{2}-b$$

$$b=\tfrac{\pi}{2}-a$$

$$\Rightarrow I = \ln\left[\frac{1+\sin(\tfrac{\pi}{2}-a)}{1+\sin(\tfrac{\pi}{2}-b)}\right] = \ln\left[\frac{1+\cos a}{1+\cos b}\right] = J$$