INTEGRATION (Q 8, PAPER 1)

2009

8 (a) Find $\int \left(6x+3+\frac{1}{x^2}\right)dx.$

(b) Evaluate (i) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 3x \sin x \; dx$ (ii) $\int_{\ln 3}^{\ln 8} e^x\sqrt{1+e^x} \; dx.$

(c) Use integration methods to establish the standard formula for the volume of a cone.

SOLUTION

8 (a)

$$\int \left(6x+3+\frac{1}{x^2}\right)dx = \int (6x+3+x^{-2})\,dx$$

$$= \frac{6x^2}{2}+3x+\frac{x^{-1}}{-1}+c$$

$$= 3x^2+3x-\frac{1}{x}+c$$

> $\int x^p\,dx = \frac{x^{p+1}}{p+1}+c$ for $p \in \mathbf{R}$ except $p \neq -1.$

8 (b) (i)

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin 3x \sin x\,dx = \tfrac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\sin 3x \sin x\,dx$$

$$= \tfrac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2x - \cos 4x)\,dx$$

$$= \tfrac{1}{2}\left[\frac{\sin 2x}{2}-\frac{\sin 4x}{4}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \tfrac{1}{4}[\sin 2x - \tfrac{1}{2}\sin 4x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \tfrac{1}{4}[(\sin 2(\tfrac{\pi}{4}) - \tfrac{1}{2}\sin 4(\tfrac{\pi}{4})) - (\sin 2(-\tfrac{\pi}{4}) - \tfrac{1}{2}\sin 4(-\tfrac{\pi}{4}))]$$

$$= \tfrac{1}{4}[\sin(\tfrac{\pi}{2}) - \tfrac{1}{2}\sin\pi - (-\sin(\tfrac{\pi}{2}) + \tfrac{1}{2}\sin\pi)]$$

$$= \tfrac{1}{4}[\sin(\tfrac{\pi}{2}) - \tfrac{1}{2}\sin\pi + \sin(\tfrac{\pi}{2}) - \tfrac{1}{2}\sin\pi]$$

$$= \tfrac{1}{4}[1-\tfrac{1}{2}(0)+1-\tfrac{1}{2}(0)]$$

$$= \tfrac{1}{4}[2]$$

$$= \tfrac{1}{2}$$

PRODUCTS $\rightarrow$ SUMS
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

> $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$

8 (b) (ii)

$$\int_{\ln 3}^{\ln 8} e^x\sqrt{1+e^x}\,dx$$
$$=\int_4^9 \sqrt{u}\,du$$
$$=\int_4^9 u^{\frac{1}{2}}\,du$$
$$=\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^9 = \tfrac{2}{3}[u^{\frac{3}{2}}]_4^9$$
$$=\tfrac{2}{3}[9^{\frac{3}{2}}-4^{\frac{3}{2}}]$$
$$=\tfrac{2}{3}[27-8]$$
$$=\tfrac{2}{3}[19]$$
$$=\tfrac{38}{3}$$

Make a substitution:

$u=1+e^x \Rightarrow du=e^x dx$

Change limits:

$u=\ln 8 \Rightarrow u=1+e^{\ln 8}=1+8=9$

$u=\ln 3 \Rightarrow u=1+e^{\ln 3}=1+3=4$

8 (c)

Slope of $L=\dfrac{r}{h} \Rightarrow$ Equation of L: $rx-hy=0 \Rightarrow y=\dfrac{r}{h}x$

Volume of cone $V=\pi\displaystyle\int_0^h y^2\,dx$

$$=\pi\frac{r^2}{h^2}\int_0^h x^2\,dx=\pi\frac{r^2}{3h^2}[x^3]_0^h=\pi\frac{r^2}{3h^2}\{h^3-0\}$$

$$=\tfrac{1}{3}\pi r^2 h$$