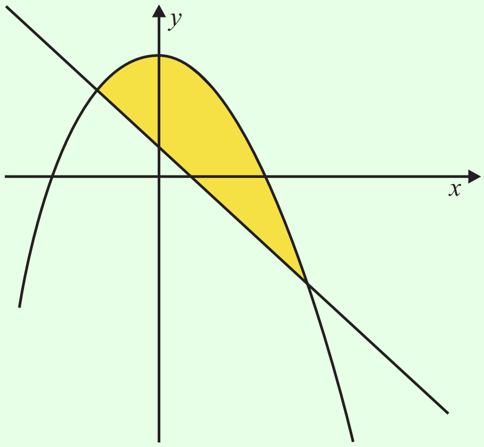
INTEGRATION (Q 8, PAPER 1)

2008

8 (a) Find $\int (2x+\cos 3x)\,dx$.

(b) Evaluate (i) $\int_0^1 3x^2 e^{x^3}\,dx$ (ii) $\int_2^4 \frac{2x^3}{x^2-1}\,dx$

(c) The diagram shows the curve $y = 4 - x^2$ and the line $2x + y - 1 = 0$.
Calculate the area of the shaded region enclosed by the curve and the line.



SOLUTION

8 (a)

$$\int (2x+\cos 3x)\,dx = \frac{2x^2}{2} + \tfrac{1}{3}\sin 3x + c$$

$$= x^2 + \tfrac{1}{3}\sin 3x + c$$

$\int x^p\,dx = \frac{x^{p+1}}{p+1} + c$ for $p \in \mathbf{R}$ except $p \neq -1$. **1**

Remember it as:

Add one to the power of x and divide by the new power.

$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$ **5**

8 (b) (i)

STEPS
1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

$\int e^x\,dx = e^x + c$ **3**

1. Let $u = x^3$

2. $du = 3x^2\,dx$

3. $x = 1 \Rightarrow u = 1$

$x = 0 \Rightarrow u = 0$

4. $I = \int_0^1 3x^2 e^{x^3}\,dx = \int_0^1 e^u\,du$

5. $I = [e^u]_0^1 = e^1 - e^0 = e - 1$

8 (b) (ii)

Steps
1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

1. Let $u = x^2 - 1 \Rightarrow x^2 = u + 1$

2. $du = 2x\,dx$

3. $x = 4 \Rightarrow u = (4)^2 - 1 = 15$

$x = 2 \Rightarrow u = (2)^2 - 1 = 3$

4. $I = \int_2^4 \frac{2x^3}{x^2-1}dx = \int_2^4 \frac{2x(x^2)}{x^2-1}dx = \int_3^{15} \frac{(u+1)}{u}du = \int_3^{15}\left(1+\frac{1}{u}\right)du$

5. $I = [u + \ln u]_3^{15} = [(15 + \ln 15) - (3 + \ln 3)]$

$= 15 - 3 + \ln 15 - \ln 3$

$= 12 + \ln \frac{15}{3}$

$= 12 + \ln 5$

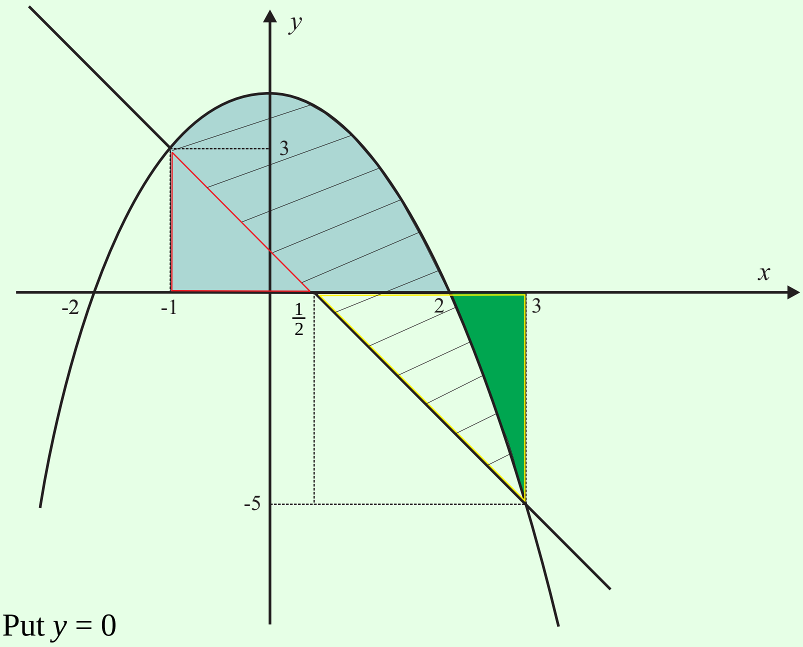
8 (c)

Draw a diagram putting in all the relevant numbers.

Curve cuts x-axis: Put $y = 0$

$0 = 4 - x^2 \Rightarrow x^2 = 4$

$\therefore x = \pm 2$



Line cuts the x-axis: Put $y = 0$

$2x + (0) - 1 = 0 \Rightarrow 2x = 1$

$\therefore x = \frac{1}{2}$

Intersection of line and curve:

Curve: $y = 4 - x^2$

Line: $2x + y - 1 = 0 \Rightarrow y = 1 - 2x$

$\therefore 1 - 2x = 4 - x^2 \Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x+1)(x-3) = 0$

$\therefore x = -1, 3$

Find the corresponding y-coordinates by substituting these values of x into the equation of the line.

$\therefore y = 3, -5$

Shaded area above the x-axis: Blue area under curve – Area of red triangle.
Remember the area of a triangle is half the base by the height.

Blue area under curve: $|A| = \int_{-1}^{2} (4 - x^2)dx = [4x - \frac{1}{3}x^3]_{-1}^{2}$

$= [(8 - \frac{8}{3}) - (-4 + \frac{1}{3})] = [8 - \frac{8}{3} + 4 - \frac{1}{3}] = 9$

Area of red triangle: $A = \frac{1}{2}(\frac{3}{2})(3) = \frac{9}{4}$

Therefore, shaded area above x-axis: $9 - \frac{9}{4} = \frac{27}{4}$

Shaded area below the x-axis: Area of yellow triangle – Green area under curve.

Area of yellow triangle: $A = \frac{1}{2}(\frac{5}{2})(5) = \frac{25}{4}$

Green area under curve: $|A| = \left|\int_{2}^{3} (4 - x^2)\,dx\right| = \left|[4x - \frac{1}{3}x^3]_{2}^{3}\right|$

$= \left|[(12 - 9) - (8 - \frac{8}{3})]\right|$

$= \left|[12 - 9 - 8 + \frac{8}{3}]\right| = \left|-\frac{7}{3}\right|$

$= \frac{7}{3}$

Therefore, shaded area below x-axis: $\frac{25}{4} - \frac{7}{3} = \frac{47}{12}$

Total shaded area: $\frac{27}{4} + \frac{47}{12} = \frac{32}{3}$

OR Here is a nice method, very slick.

Where two curves, $f(x)$ and $g(x)$, overlap the area between the curves is given by

$\left|\int_{a}^{b} (f(x) - g(x))\,dx\right|$, where a and b are the limits between the overlap.

Area of overlap: $A = \int_{-1}^{3} (4 - x^2 - 1 + 2x)\,dx = \int_{-1}^{3} (3 + 2x - x^2)\,dx$

$\Rightarrow A = \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3} = \left[\left(9 + 9 - \frac{27}{3}\right) - \left(-3 + 1 + \frac{1}{3}\right)\right]$

$\Rightarrow A = [(9) - (-\frac{5}{3})]$

$\therefore A = \frac{32}{3}$