INTEGRATION (Q 8, PAPER 1)

2000

8 (a) Find (i) $\int (x^2+2)dx$ (ii) $\int e^{3x}dx.$

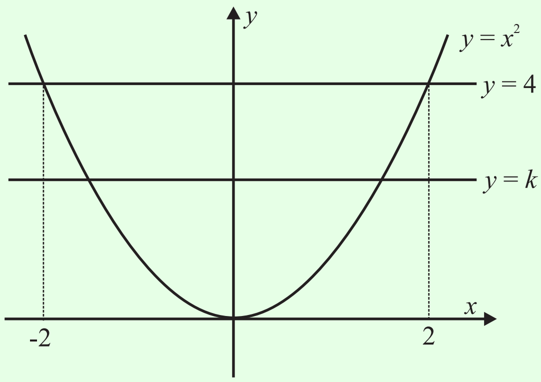
8 (b) Evaluate (i) $\int_0^{\frac{\pi}{2}} \sin^2 3\theta \, d\theta$ (ii) $\int_0^1 \frac{x}{x^2+4}dx.$

8 (c) (i) Find the value of the real number p given that

$$\int_2^p \frac{dx}{x^2-4x+5} = \frac{\pi}{4}.$$

(ii) The region bounded by the curve $y = x^2$ and the line $y = 4$ is divided into two regions of equal area by the line $y = k$.

Show that $k^3 = 16$.



SOLUTION

8 (a) (i)

$$\int (x^2+2)dx = \frac{x^3}{3} + 2x + c$$

$\int x^p dx = \frac{x^{p+1}}{p+1} + c$ for $p \in \mathbf{R}$ except $p \neq -1$. **1**

Remember it as:

Add one to the power of x and divide by the new power.

8 (a) (ii)

$$\int e^{3x}dx = \tfrac{1}{3}e^{3x} + c$$

$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$ **3**

8 (b) (i)

$$\int_0^{\frac{\pi}{2}} \sin^2 3\theta \, d\theta = \tfrac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 6\theta)\, d\theta$$

$\sin^2 A = \frac{1}{2}(1-\cos 2A)$

$\cos^2 A = \frac{1}{2}(1+\cos 2A)$

$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{2}[(\tfrac{\pi}{2} - \tfrac{1}{6}\sin 6(\tfrac{\pi}{2})) - (0 - \tfrac{1}{6}\sin 6(0))]$$

$$= \tfrac{1}{2}[(\tfrac{\pi}{2} - 0) - (0-0)] = \tfrac{\pi}{4}$$

8 (b) (ii)

STEPS
1. Let *u* equal the *inside* of the more complicated function.
2. Differentiate *u* with respect to *x*.
3. Change the limits from *x* to *u*.
4. Make the substitution.
5. Evaluate the integral *I*.

1. Let $u = x^2 + 4$.

2. $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

3. $x = 1 \Rightarrow u = (1)^2 + 4 = 5$

$x = 0 \Rightarrow u = (0)^2 + 4 = 4$

4. $I = \int_0^1 \frac{x}{x^2+4}dx = \frac{1}{2}\int_4^5 \frac{du}{u}x$ $\quad \boxed{\int \frac{1}{x}dx = \ln x + c}$ **2**

5. $\therefore I = \frac{1}{2}[\ln u]_4^5 = \frac{1}{2}[\ln 5 - \ln 4]$

$\therefore I = \frac{1}{2}\ln(\frac{5}{4})$

8 (c) (i)

$$\int \frac{dx}{(a)^2 + (x \pm b)^2} = \frac{1}{a}\tan^{-1}\left(\frac{x \pm b}{a}\right) + c$$ **6**

$$\int_2^p \frac{dx}{x^2 - 4x + 5} = \frac{\pi}{4}$$

$$\Rightarrow \int_2^p \frac{dx}{(1)^2 + (x-2)^2} = \frac{\pi}{4}$$

$$\Rightarrow [\tan^{-1}(x-2)]_2^p = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}(p-2) - \tan^{-1}0 = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}(p-2) = \frac{\pi}{4}$$

$$\Rightarrow p - 2 = \tan(\tfrac{\pi}{4}) = 1$$

$$\therefore p = 3$$

$x^2 - 4x + 5$

$= x^2 - 4x + 4 + 1$

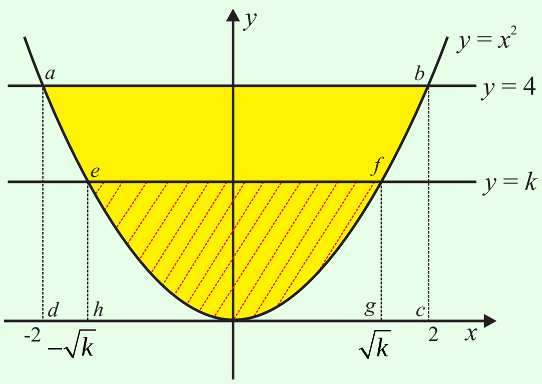
$= (x-2)^2 + 1^2$

8 (c) (ii)

Firstly, find where the line $y = k$ and the curve intersect.

$y = x^2 \Rightarrow k = x^2$

$\therefore x = \pm\sqrt{k}$



Find the area of the yellow region.

Area = Area of rectangle *abcd* – Area under curve from *a* to *b*.

$\therefore$ Area $= (4)(4) - \int_{-2}^{2} x^2 dx = 16 - \frac{1}{3}[x^3]_{-2}^{2}$

$\Rightarrow$ Area $= 16 - \frac{1}{3}[(2)^3 - (-2)^3]$

$\Rightarrow$ Area $= 16 - \frac{1}{3}[8 - (-8)]$

$\Rightarrow$ Area $= 16 - \frac{16}{3}$

$\therefore$ Area $= \frac{32}{3}$

Area of Red Shaded region $= \frac{16}{3}$

Area of Red Shaded region = Area of rectangle *efgh* – Area under curve from *e* to *f*

$\Rightarrow$ Area $= 2k\sqrt{k} - \int_{-\sqrt{k}}^{\sqrt{k}} x^2 dx = \frac{16}{3}$

$\Rightarrow 2k\sqrt{k} - \frac{1}{3}[x^3]_{-\sqrt{k}}^{\sqrt{k}} = \frac{16}{3}$

$\Rightarrow 2k^{\frac{3}{2}} - \frac{1}{3}[(\sqrt{k})^3 - (-\sqrt{k})^3] = \frac{16}{3}$

$\Rightarrow 2k^{\frac{3}{2}} - \frac{1}{3}[k^{\frac{3}{2}} - (-k^{\frac{3}{2}})] = \frac{16}{3}$

$\Rightarrow 2k^{\frac{3}{2}} - \frac{2}{3}[k^{\frac{3}{2}}] = \frac{16}{3}$

$\Rightarrow \frac{4}{3}[k^{\frac{3}{2}}] = \frac{16}{3}$

$\Rightarrow k^{\frac{3}{2}} = 4$ [Square both sides.]

$\therefore k^3 = 16$