INTEGRATION (Q 8, PAPER 1)

1999

8 (a) Find $\int \left(4x+1+\frac{1}{x^3}\right)dx.$

(b) Evaluate (i) $\int_0^{\frac{\pi}{6}} 2\cos 4\theta \cos 2\theta \, d\theta$ (ii) $\int_{-3}^{0} (x+3)e^{x(x+6)}\, dx.$

(c) Evaluate $\int_0^{\sqrt{3}} \sqrt{4-x^2}\, dx.$

Hint: Let $x = 2\sin\theta.$

SOLUTION

8 (a)

$$\int \left(4x+1+\frac{1}{x^3}\right)dx = \int (4x+1+x^{-3})\, dx$$

$$= \frac{4x^2}{2} + x + \frac{x^{-2}}{-2} + c$$

$$= 2x^2 + x - \frac{1}{2x^2} + c$$

$\int x^p dx = \frac{x^{p+1}}{p+1} + c$ for $p \in \mathbf{R}$ except $p \neq -1.$ **1**

Remember it as:

Add one to the power of x and divide by the new power.

8 (b) (i)

$$I = \int_0^{\frac{\pi}{6}} 2\cos 4\theta \cos 2\theta \, d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{6}} (\cos 6\theta + \cos 2\theta)\, d\theta$$

$$\Rightarrow I = [\tfrac{1}{6}\sin 6\theta + \tfrac{1}{2}\sin 2\theta]_0^{\frac{\pi}{6}}$$

$$\Rightarrow I = [(\tfrac{1}{6}\sin 6(\tfrac{\pi}{6}) + \tfrac{1}{2}\sin 2(\tfrac{\pi}{6})) - (\tfrac{1}{6}\sin 0 + \tfrac{1}{2}\sin 0)]$$

$$\Rightarrow I = [(\tfrac{1}{6}\sin \pi + \tfrac{1}{2}\sin(\tfrac{\pi}{3})) - (\tfrac{1}{6}\sin 0 + \tfrac{1}{2}\sin 0)]$$

$$\Rightarrow I = [(0 + \tfrac{1}{2}(\tfrac{\sqrt{3}}{2})) - (0)]$$

$$\therefore I = \frac{\sqrt{3}}{4}$$

PRODUCTS $\longrightarrow$ SUMS
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$
$2\sin A\sin B = \cos(A-B) - \cos(A+B)$
$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

8 (b) (ii)

Steps
1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

1. Let $u = x^2 + 6x$
2. $du = (2x+6)\,dx = 2(x+3)\,dx$

   $\Rightarrow \frac{1}{2}du = (x+3)\,dx$
3. $x = 0 \Rightarrow u = (0)^2 + 6(0) = 0$

   $x = -3 \Rightarrow u = (-3)^2 + 6(-3) = 9 - 18 = -9$
4. $I = \frac{1}{2}\int_{-9}^{0} e^u\,du$ $\quad$ $\int e^x dx = e^x + c$ **3**
5. $I = \frac{1}{2}[e^u]_{-9}^{0} = \frac{1}{2}[e^0 - e^{-9}]$

   $\therefore I = \frac{1}{2}\left(1 - \frac{1}{e^9}\right)$

8 (c)

$$I = \int_0^{\sqrt{3}} \sqrt{4 - x^2}\,dx$$

1. Let $x = 2\sin\theta$
2. $dx = 2\cos\theta\,d\theta$
3. $x = \sqrt{3} \Rightarrow \sqrt{3} = 2\sin\theta$

   $\Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$

   $x = 0 \Rightarrow 0 = 2\sin\theta$

   $\Rightarrow \sin\theta = 0 \Rightarrow \theta = \sin^{-1} 0 = 0$
4. $I = \int_0^{\frac{\pi}{3}} \sqrt{4 - 4\sin^2\theta}\;2\cos\theta\,d\theta$

   $\Rightarrow I = \int_0^{\frac{\pi}{3}} \sqrt{4(1-\sin^2\theta)}\;2\cos\theta\,d\theta$ $\quad$ $\cos^2 A + \sin^2 A = 1$ **8**

   $\Rightarrow I = \int_0^{\frac{\pi}{3}} 2\sqrt{\cos^2\theta}\;2\cos\theta\,d\theta$

   $\Rightarrow I = 4\int_0^{\frac{\pi}{3}} \cos^2\theta\,d\theta$ $\quad$ $\cos^2 A = \frac{1}{2}(1+\cos 2A)$

   $\Rightarrow I = \frac{4}{2}\int_0^{\frac{\pi}{3}} (1+\cos 2\theta)\,d\theta$

   $\therefore I = 2\int_0^{\frac{\pi}{3}} (1+\cos 2\theta)\,d\theta$ $\quad$ $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$ **5**

5. $I = 2[\theta + \frac{1}{2}\sin 2\theta]_0^{\frac{\pi}{3}}$

$\Rightarrow I = 2[(\frac{\pi}{3} + \frac{1}{2}\sin 2(\frac{\pi}{3})) - (0 + \frac{1}{2}\sin 0)]$

$\Rightarrow I = 2[(\frac{\pi}{3} + \frac{1}{2}(\frac{\sqrt{3}}{2})) - (0)]$

$\therefore I = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$