1998

8 (a) Find (i) $\int (x^2+3)\,dx$ (ii) $\int \frac{1}{x^2}\,dx$.

(b) Evaluate (i) $\int_2^3 \frac{x-2}{x^2-4x+5}\,dx$ (ii) $\int_0^{\frac{\pi}{4}} (\cos x+\sin x)^2\,dx$

(c) Find the area of the bounded region enclosed by the line $y=2x-1$, the line $x = 4$ and the curve $y=\frac{1}{x}$, where $x > 0$.

SOLUTION

8 (a) (i)

$$\int (x^2+3)\,dx = \tfrac{1}{3}x^3+3x+c$$

$\int x^p\,dx = \frac{x^{p+1}}{p+1}+c$ for $p \in \mathbf{R}$ except $p \neq -1$. **1**

Remember it as:

Add one to the power of x and divide by the new power.

8 (a) (ii)

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1}+c = -\tfrac{1}{x}+c$$

8 (b) (i)

$$I = \int_2^3 \frac{x-2}{x^2-4x+5}\,dx$$

1. Let $u = x^2-4x+5 \Rightarrow du = (2x-4)dx$

2. $du = 2(x-2)dx$

$\Rightarrow \tfrac{1}{2}du = (x-2)dx$

3. $x=3 \Rightarrow u=(3)^2-4(3)+5 = 9-12+5 = 2$

$x=2 \Rightarrow u=(2)^2-4(2)+5 = 4-8+5 = 1$

4. $I = \tfrac{1}{2}\int_1^2 \frac{du}{u}$

$\int \frac{1}{x}\,dx = \ln x + c$ **2**

5. $\therefore I = \tfrac{1}{2}[\ln u]_1^2 = \tfrac{1}{2}[\ln 2 - \ln 1] = \tfrac{1}{2}\ln 2$

8 (b) (ii)

$$I = \int_0^{\frac{\pi}{4}} (\cos x + \sin x)^2 dx$$

$\cos^2 A + \sin^2 A = 1$ 8

$\sin 2A = 2\sin A \cos A$ 13

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} (\cos^2 x + 2\cos x \sin x + \sin^2 x)\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} (1 + \sin 2x)\, dx$$

$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$ 4

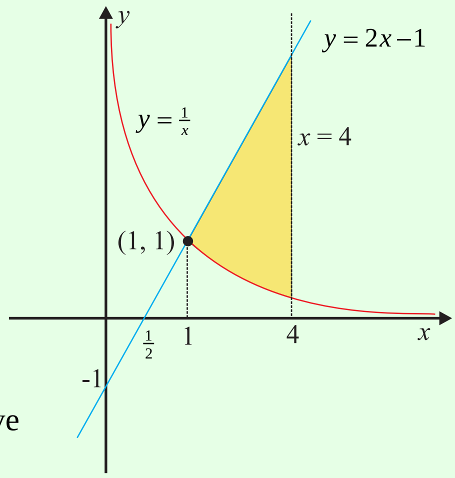
$$\Rightarrow I = [x - \tfrac{1}{2}\cos 2x]_0^{\frac{\pi}{4}} = [(\tfrac{\pi}{4} - \tfrac{1}{2}\cos 2(\tfrac{\pi}{4})) - (0 - \tfrac{1}{2}\cos 0)]$$

$$\Rightarrow I = [(\tfrac{\pi}{4} - \tfrac{1}{2}(0)) - (0 - \tfrac{1}{2}(1))]$$

$$\therefore I = \tfrac{\pi}{4} + \tfrac{1}{2}$$

8 (c)

The key to doing this problem is to draw the 3 curves.
x = 4: Draw a vertical line through x = 4 on the x-axis.



$y = 2x - 1$: $(0, -1)$, $(\frac{1}{2}, 0)$

$y = \frac{1}{x}$: This is a rational curve. It has asymtotes of x = 0 and y = 0. It comes in 2 halves but you are only interested in the half in the first quadrant where $x > 0$.

You need to find out where the blue line and the red curve intersect.

$$y = 2x - 1 \Rightarrow \tfrac{1}{x} = 2x - 1$$

$$\Rightarrow 1 = 2x^2 - x$$

$$\Rightarrow 2x^2 - x - 1 = 0$$

$$\Rightarrow (2x+1)(x-1) = 0$$

$$\therefore x = -\tfrac{1}{2}, 1$$

$$\therefore y = -2, 1$$

$\therefore$ (1, 1) is the point on intersection in the first quadrant.

Area of yellow part = Area under blue line from 1 to 4 – Area under red curve from 1 to 4

$$\therefore A = \int_1^4 (2x-1)\, dx - \int_1^4 \frac{1}{x} dx$$

$$\Rightarrow A = [x^2 - x]_1^4 - [\ln x]_1^4$$

$$\Rightarrow A = [(16-4) - (1-1)] - [\ln 4 - \ln 1]$$

$$\therefore A = 12 - \ln 4$$