INTEGRATION (Q 8, PAPER 1)

1997

8 (a) Find (i) $\int \sin 4x\, dx$ (ii) $\int (1+\sqrt{x})^2\, dx.$

(b) Evaluate (i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2 3\theta\, d\theta$ (ii) $\int_0^1 \frac{x^2}{x+1}\, dx.$

(c) Calculate the value of

$$\int_{\frac{1}{3}}^{3} \frac{1}{t+\sqrt{t}}\, dt.$$

Hint: $u = \sqrt{t}.$

SOLUTION

8 (a) (i)

$$\int \sin 4x\, dx = -\tfrac{1}{4}\cos 4x + c$$

$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+c$ 4

8 (a) (ii)

$$\int (1+\sqrt{x})^2\, dx = \int (1+2\sqrt{x}+x)\, dx$$

$$= \int (1+2x^{\frac{1}{2}}+x)\, dx$$

$$= x + \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{2} + c$$

$$= x + \tfrac{4}{3}x^{\frac{3}{2}} + \tfrac{1}{2}x^2 + c$$

$\int x^p dx = \frac{x^{p+1}}{p+1} + c$ for $p \in \mathbf{R}$ except $p \neq -1.$ 1

Remember it as:

Add one to the power of x and divide by the new power.

8 (b) (i)

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2 3\theta\, d\theta$$

$\cos^2 A = \frac{1}{2}(1+\cos 2A)$

$$\Rightarrow I = \tfrac{1}{2}(2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 6\theta)\, d\theta$$

$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$ 5

$$\therefore I = [\theta + \tfrac{1}{6}\sin 6\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\Rightarrow I = [(\tfrac{\pi}{2}+\tfrac{1}{6}\sin 6(\tfrac{\pi}{2})) - (-\tfrac{\pi}{2}+\tfrac{1}{6}\sin 6(-\tfrac{\pi}{2}))]$$

$$\Rightarrow I = [(\tfrac{\pi}{2}+\tfrac{1}{6}\sin 3\pi) - (-\tfrac{\pi}{2}-\tfrac{1}{6}\sin 3\pi)]$$

$$\therefore I = \tfrac{\pi}{2}+\tfrac{\pi}{2} = \pi$$

8 (b) (ii)

STEPS
1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

$$I=\int_0^1 \frac{x^2}{x+1}dx$$

1. Let $u=x+1 \Rightarrow x=(u-1)$

2. $\Rightarrow du=dx$

3. $x=1 \Rightarrow u=1+1=2$
$x=0 \Rightarrow u=0+1=1$

4. $I=\int_1^2 \frac{(u-1)^2}{u}du=\int_1^2 \frac{(u^2-2u+1)}{u}du=\int_1^2 (u-2+\frac{1}{u})\,du$ $\int \frac{1}{x}dx=\ln x+c$ **2**

5. $\therefore I=[\frac{1}{2}u^2-2u+\ln u]_1^2=[(2-4+\ln 2)-(\frac{1}{2}-2+\ln 1)]$
$\therefore I=[-2+\ln 2-\frac{1}{2}+2+0]$
$\therefore I=\ln 2-\frac{1}{2}$

8 (c)

$$I=\int_{\frac{1}{3}}^3 \frac{1}{t+\sqrt{t}}dt=\int_{\frac{1}{3}}^3 \frac{1}{\sqrt{t}(\sqrt{t}+1)}dt$$

1. Let $u=\sqrt{t}=t^{\frac{1}{2}}$

2. $\Rightarrow du=\frac{1}{2}t^{-\frac{1}{2}}dt=\frac{1}{2\sqrt{t}}dt$
$\Rightarrow 2du=\frac{dt}{\sqrt{t}}$

3. $x=3 \Rightarrow u=\sqrt{3}$
$x=\frac{1}{3} \Rightarrow u=\frac{1}{\sqrt{3}}$

4. $I=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{2du}{(u+1)}=2\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{(u+1)}du$

5. $\therefore I = 2[\ln(u+1)]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

$$\Rightarrow I = 2[\ln(\sqrt{3}+1) - \ln(\tfrac{1}{\sqrt{3}}+1)]$$

$$\Rightarrow I = 2\ln\left(\frac{\sqrt{3}+1}{\frac{1}{\sqrt{3}}+1}\right)$$

$$\therefore I = 2\ln\sqrt{3} = \ln(\sqrt{3})^2 = \ln 3$$