1996

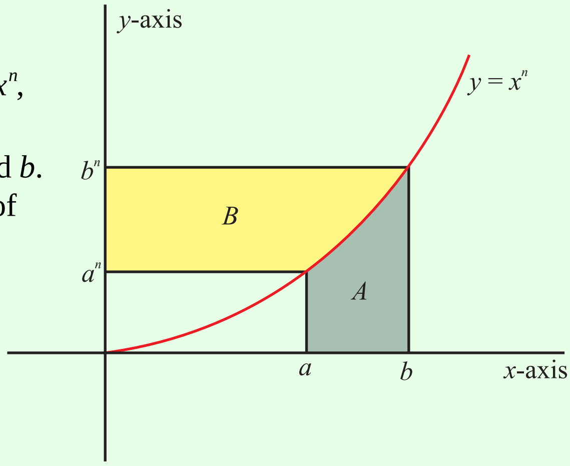
8 (a) (i) $\int \frac{1}{x^2}\,dx$ (ii) $\int (2x-1)^2\,dx$.

(b) (i) $\int_0^2 \frac{dt}{\sqrt{4-t^2}}$ (ii) $\int_0^{\frac{\pi}{3}} \sin 2\theta \cos\theta \, d\theta$.

(c) (i) Calculate $\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$ to three places of decimals.

(ii) A is the area between the curve $y = x^n$, the x-axis and the lines $x = a$, $x = b$. Calculate the area A in terms of a and b. B is the area between the same part of the curve and the y-axis. Determine the ratio

Area B: Area A.



SOLUTION

8 (a) (i)

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx$$

$$= \frac{x^{-1}}{-1} + c$$

$$= -\tfrac{1}{x} + c$$

$\int x^p\,dx = \frac{x^{p+1}}{p+1} + c$ for $p \in \mathbf{R}$ except $p \neq -1$. **1**

Remember it as:

Add one to the power of x and divide by the new power.

8 (a) (ii)

$$\int (2x-1)^2\,dx = \int (4x^2 - 4x + 1)\,dx$$

$$= \frac{4x^3}{3} - \frac{4x^2}{2} + x + c$$

$$= \tfrac{4}{3}x^3 - 2x^2 + x + c$$

8 (b) (i)

$$I = \int_0^2 \frac{dt}{\sqrt{4-t^2}} = \int_0^2 \frac{dt}{\sqrt{2^2-t^2}}$$

$$\int \frac{dx}{\sqrt{(a)^2-(x)^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c$$ 7

$$\therefore I = \left[\sin^{-1}\left(\frac{t}{2}\right)\right]_0^2$$

$$\therefore I = [\sin^{-1}1 - \sin^{-1}0] = \tfrac{\pi}{2}$$

8 (b) (ii)

$$I = \int_0^{\frac{\pi}{3}} \sin 2\theta \cos\theta \, d\theta$$

PRODUCTS → SUMS
$2\sin A\cos B = \sin(A+B)+\sin(A-B)$

$$\Rightarrow I = \tfrac{1}{2}\int_0^{\frac{\pi}{3}} (\sin 3\theta + \sin\theta)\, d\theta$$

$$\int \sin(ax+b)dx = -\tfrac{1}{a}\cos(ax+b)+c$$ 4

$$\therefore I = \tfrac{1}{2}[-\tfrac{1}{3}\cos 3\theta - \cos\theta]_0^{\frac{\pi}{3}}$$

$$\Rightarrow I = \tfrac{1}{2}[(-\tfrac{1}{3}\cos\pi - \cos\tfrac{\pi}{3}) - (-\tfrac{1}{3}\cos 0 - \cos 0)]$$

$$\Rightarrow I = \tfrac{1}{2}[-\tfrac{1}{3}(-1) - \tfrac{1}{2} + \tfrac{1}{3} + 1]$$

$$\Rightarrow I = \tfrac{1}{2}[\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{3} + 1] = \tfrac{1}{2}[\tfrac{7}{6}]$$

$$\therefore I = \tfrac{7}{12}$$

8 (c) (i)

STEPS
1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

$$I = \int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\, dx$$

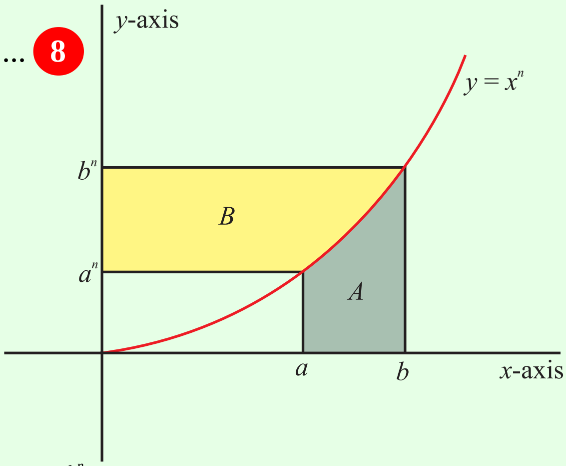
1. Let $u = e^x$
2. $\Rightarrow du = e^x dx$
3. $x = \ln\sqrt{3} \Rightarrow u = e^{\ln\sqrt{3}} = \sqrt{3}$ [When e-ln come together, they cancel.]

   $x = 0 \Rightarrow u = e^0 = 1$
4. $I = \displaystyle\int_1^{\sqrt{3}} \frac{du}{1+u^2}$

   $$\int \frac{dx}{(a)^2+(x)^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$ 6
5. $\therefore I = [\tan^{-1}u]_1^{\sqrt{3}} = [\tan^{-1}\sqrt{3} - \tan^{-1}1]$

   $\therefore I = \tfrac{\pi}{3} - \tfrac{\pi}{4} = \tfrac{\pi}{12} = 0.262$

8 (c) (ii)



$$\text{Area of } A = \int_a^b x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

$$\boxed{A = \int_a^b y\,dx} \quad \ldots\ldots\ 8$$

$$\Rightarrow \text{Area of } A = \left[\frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}\right]$$

$$\Rightarrow \text{Area of } A = \frac{b^{n+1} - a^{n+1}}{n+1}$$

$$y = x^n \Rightarrow x = y^{\frac{1}{n}}$$

$$\boxed{A = \int_a^b x\,dy} \quad \ldots\ldots\ 8$$

$$\text{Area of } B = \int_{a^n}^{b^n} y^{\frac{1}{n}}\,dy = \left[\frac{y^{\frac{1}{n}+1}}{\frac{1}{n}+1}\right]_{a^n}^{b^n} = \left[\frac{y^{\frac{n+1}{n}}}{\frac{n+1}{n}}\right]_{a^n}^{b^n} = \left[\frac{ny^{\frac{n+1}{n}}}{n+1}\right]_{a^n}^{b^n}$$

$$\Rightarrow \text{Area of } B = \left[\frac{nb^{n\left(\frac{n+1}{n}\right)}}{n+1} - \frac{na^{n\left(\frac{n+1}{n}\right)}}{n+1}\right]_{a^n}^{b^n}$$

$$\Rightarrow \text{Area of } B = \frac{n(b^{n+1} - a^{n+1})}{n+1}$$

$$\therefore \text{Area of } A : \text{Area of } B = \frac{n(b^{n+1} - a^{n+1})}{n+1} : \frac{b^{n+1} - a^{n+1}}{n+1} = n : 1$$