INTEGRATION (Q 8, PAPER 1)

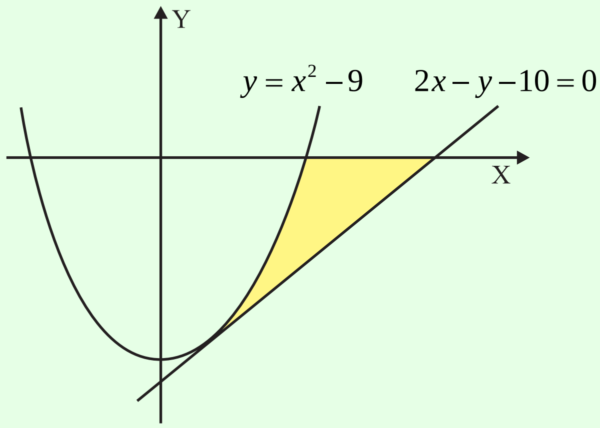
2007

8 (a) Find (i) $\int x^3 dx$ (ii) $\int \frac{1}{x^3} dx.$

(b) (i) Evaluate $\int_0^4 x\sqrt{x^2+9}\, dx.$

(ii) f is a function such that $f'(x) = 6 - \sin x$ and $f(\frac{\pi}{3}) = 2\pi.$
Find $f(x)$.

(c) The line $2x - y - 10 = 0$ is a tangent to the curve $y = x^2 - 9$, as shown.
The shaded region is bounded by the line, the curve and the x-axis.
Calculate the area of this region.



SOLUTION

$$\int x^p dx = \frac{x^{p+1}}{p+1} + c \text{ for } p \in \mathbf{R} \text{ except } p \neq -1. \quad \text{.......} \ \textbf{1}$$

Remember it as:

Add one to the power of x and divide by the new power.

8 (a) (i)

$$\int x^3 dx = \tfrac{1}{4}x^4 + c$$

8 (a) (ii)

$$\int \frac{1}{x^3} dx = \int x^{-3}\, dx = -\tfrac{1}{2}x^{-2} + c = -\frac{1}{2x^2} + c$$

8 (b) (i)

$$I = \int_0^4 x\sqrt{x^2+9}\,dx$$

STEPS
1. Let u equal the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

1. Let $u = x^2 + 9$

2. $\Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

3. $x = 4 \Rightarrow u = 25$

$x = 0 \Rightarrow u = 9$

4. $I = \frac{1}{2}\int_9^{16} \sqrt{u}\,du = \frac{1}{2}\int_9^{16} u^{\frac{1}{2}}\,du$

5. $I = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_9^{25} = \frac{1}{3}[u^{\frac{3}{2}}]_9^{25} = \frac{1}{3}[25^{\frac{3}{2}} - 9^{\frac{3}{2}}] = \frac{1}{3}[125 - 27] = \frac{98}{3}$

8 (b) (ii)

$f(x) = \int (6 - \sin x)\,dx = 6x + \cos x + c$

$\int \sin x\,dx = -\cos x + c$ **4**

$f(\frac{\pi}{3}) = 2\pi \Rightarrow 6(\frac{\pi}{3}) + \cos(\frac{\pi}{3}) + c = 2\pi \Rightarrow 2\pi + \frac{1}{2} + c = 2\pi$

$\therefore c = -\frac{1}{2} \Rightarrow f(x) = 6x + \cos x - \frac{1}{2}$

8 (c)

$A = \int_a^b y\,dx$ **8**

TWO CURVE PROBLEMS
1. Find the points of intersection of the curves by solving simultaneously.
2. The area A between the curve C_1 and the curve C_2 is given by:
   A = Area under C_1 − Area under C_2

1. Point of intersection of the line L: $2x - y - 10 = 0$ and the curve C: $y = x^2 - 9$.

L: $y = 2x - 10$

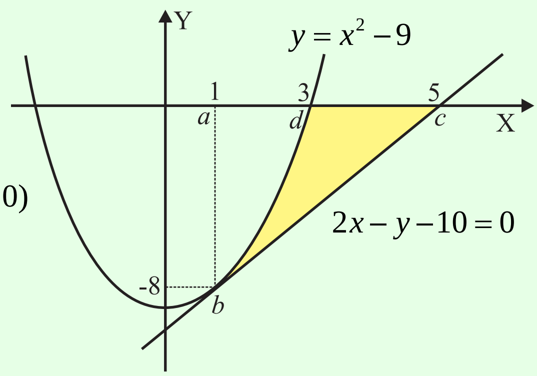
Substitute into C: $\therefore 2x - 10 = x^2 - 9 \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0 \Rightarrow x = 1$

Point of intersection of L and C: $(1, -8)$

Find out where L and C cross the X-axis:

L: Put $y = 0 \Rightarrow 2x = 10 \Rightarrow x = 5 \Rightarrow (5, 0)$ is the X-intercept.

C: Put $y = 0 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3 \Rightarrow (-3, 0), (3, 0)$ are the X-intercepts.



2. Shaded Area = |Area $\triangle abc$| − |Area abd|

|Area $\triangle abc$| $= \frac{1}{2}(4)(8) = 16$

|Area abd| $= \int_1^3 (x^2 - 9)\,dx = [\frac{1}{3}x^3 - 9x]_1^3 = [(9 - 27) - (\frac{1}{3} - 9)] = |-\frac{28}{3}| = \frac{28}{3}$

Shaded area $= 16 - \frac{28}{3} = \frac{20}{3}$