INTEGRATION (Q 8, PAPER 1)

2006

8 (a) Find (i) $\int \sqrt{x}dx$ (ii) $\int e^{-2x}dx$.

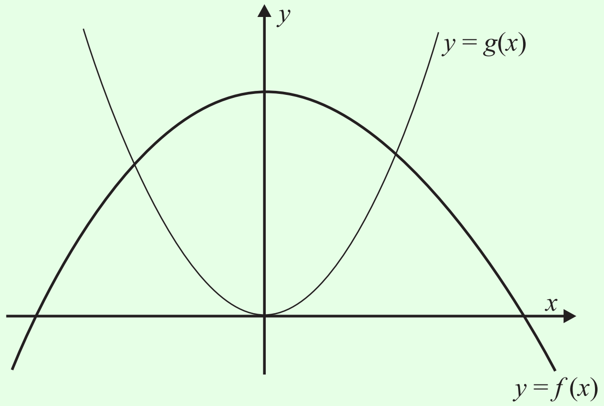
8 (b) Evaluate (i) $\int_1^2 x(1+x^2)^3 dx$ (ii) $\int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta$.

8 (c) The diagram shows the graphs of the curves $y = f(x)$ and $y = g(x)$, where

$f(x) = 12 - 3x^2$ and $g(x) = 9x^2$.

(i) Calculate the area of the region enclosed by the curve $y = f(x)$ and the x-axis.

(ii) Show that the region enclosed by the curves $y = f(x)$ and $y = g(x)$ has half that area.



SOLUTION

8 (a) (i)

$$\int x^p dx = \frac{x^{p+1}}{p+1} + c \text{ for } p \in \mathbf{R} \text{ except } p \neq -1. \quad \text{.......} \ \mathbf{1}$$

Remember it as:

Add one to the power of x and divide by the new power.

$$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \tfrac{2}{3} x^{\frac{3}{2}} + c$$

8 (a) (ii)

$$\int e^{ax+b} dx = \tfrac{1}{a} e^{ax+b} + c \quad \text{.......} \ \mathbf{3}$$

$$\int e^{-2x} dx = -\tfrac{1}{2} e^{-2x} + c$$

8 (b) (i)

$$I = \int_1^2 x(1+x^2)\, dx$$

STEPS
1. Let u equal the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

1. Let $u = 1 + x^2$.

2. $du = 2x\, dx \Rightarrow \tfrac{1}{2} du = x\, dx$

3. $x = 1 \Rightarrow u = 1 + (1)^2 = 2$ and $x = 2 \Rightarrow u = 1 + (2)^2 = 5$

4. $I = \tfrac{1}{2} \int_2^5 u^3 du$

5. $\therefore I = \tfrac{1}{2}[\tfrac{1}{4}u^4]_2^5 = \tfrac{1}{8}[u^4]_2^5 = \tfrac{1}{8}[625 - 16] = \tfrac{609}{8}$

8 (b) (ii)

$$I = \int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta$$

PRODUCTS $\longrightarrow$ SUMS
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$
$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

STEPS
1. Change products into sums using the formulae on page 9.
2. Integrate each part.

1. $I = \int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin 8\theta + \sin 2\theta)\, d\theta$

$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$ **4**

2. $I = \int_0^{\frac{\pi}{4}} \sin 5\theta \cos 3\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin 8\theta + \sin 2\theta)\, d\theta$

$$= \tfrac{1}{2}[-\tfrac{1}{8}\cos 8\theta - \tfrac{1}{2}\cos 2\theta]_0^{\frac{\pi}{4}} = -\tfrac{1}{4}[\tfrac{1}{4}\cos 8\theta + \cos 2\theta]_0^{\frac{\pi}{4}}$$

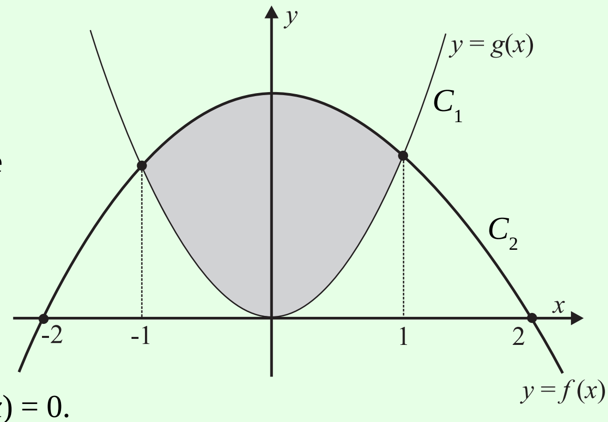
$$= -\tfrac{1}{4}[(\tfrac{1}{4}\cos 2\pi + \cos \tfrac{\pi}{2}) - (\tfrac{1}{4}\cos 0 + \cos 0)]$$

$$= -\tfrac{1}{4}[(\tfrac{1}{4}(1) + 0) - (\tfrac{1}{4}(1) + 1)] = \tfrac{1}{4}$$

8 (c) (i)

To find the area between the curve $y = f(x)$ and the x-axis, you need to find where the curve intersects the x-axis and then integrate the curve between these limits.

$A = \int_a^b y \, dx$ **8**



Intersection of the curve with x-axis: Set $f(x) = 0$.

$12 - 3x^2 = 0 \Rightarrow 12 = 3x^2 \Rightarrow x = \pm 2$

Area under C_1:

$$\int_{-2}^{2} (12 - 3x^2)\, dx = [12x - x^3]_{-2}^{2} = [(12 \times 2 - 2^3) - (12(-2) - (-2)^3)]$$

$$= [(24 - 8) - (-24 + 8)] = [(16) - (-16)] = 32$$

8 (c) (ii)

1. $f(x) = g(x) \Rightarrow 12 - 3x^2 = 9x^2$

$\Rightarrow 12 = 12x^2 \Rightarrow x = \pm 1$

2. A = Area under C_1 − Area under C_2

$$\Rightarrow A = \int_{-1}^{1} (12 - 3x^2)\, dx - \int_{-1}^{1} 9x^2 \, dx$$

$$= [12x - x^3]_{-1}^{1} - [3x^3]_{-1}^{1} = [9 - (-11)] - [3 - (-3)] = 20 - 6 = 16$$

TWO CURVE PROBLEMS
1. Find the points of intersection of the curves by solving simultaneously.
2. The area A between the curve C_1 and the curve C_2 is given by:
   A = Area under C_1 − Area under C_2