INTEGRATION (Q 8, PAPER 1)

2005

8 (a) Find (i) $\int (2+x^3)dx$ (ii) $\int e^{3x}dx$

8 (b) Evaluate (i) $\int_1^4 \frac{2x+1}{x^2+x+1}dx$ (ii) $\int_0^{\frac{\pi}{8}} \sin^2 2\theta \, d\theta$

8 (c) (i) Evaluate $\int_1^2 \frac{1}{\sqrt{3+2x-x^2}}dx.$

(ii) Use integration methods to derive a formula for the volume of a cone.

SOLUTION

8 (a) (i)

$$\int x^p dx = \frac{x^{p+1}}{p+1} + c \text{ for } p \in \mathbf{R} \text{ except } p \neq -1. \quad \text{.......} \ \mathbf{1}$$

Remember it as:

Add one to the power of *x* and divide by the new power.

$$\int (2+x^3)dx = 2x + \tfrac{1}{4}x^4 + c$$

8 (a) (ii)

$$\int e^{ax+b}dx = \tfrac{1}{a}e^{ax+b} + c \quad \text{.......} \ \mathbf{3}$$

$$\int e^{3x}dx = \tfrac{1}{3}e^{3x} + c$$

8 (b) (i)

STEPS
1. Let *u* equal the *inside* of the more complicated function.
2. Differentiate *u* with respect to *x*.
3. Change the limits from *x* to *u*.
4. Make the substitution.
5. Evaluate the integral *I*.

$$I = \int_1^4 \frac{2x+1}{x^2+x+1}dx$$

1. Let $u = x^2 + x + 1$.

2. $\Rightarrow du = (2x+1)dx$

3. $x = 1 \Rightarrow u = 1^2 + 1 + 1 = 3$ and $x = 4 \Rightarrow u = 4^2 + 4 + 1 = 21$.

4. $\therefore I = \int_7^{21} \frac{du}{u}$

$$\int \frac{1}{x}dx = \ln x + c \quad \text{.......} \ \mathbf{2}$$

5. $\Rightarrow I = [\ln u]_7^{21} = \ln 21 - \ln 7 = \ln \tfrac{21}{7} = \ln 3$

8 (b) (ii)

$$I = \int_0^{\frac{\pi}{8}} \sin^2 2\theta \, d\theta$$

Trig squares are dealt with by using these formulae:

$\sin^2 A = \frac{1}{2}(1-\cos 2A)$

$\cos^2 A = \frac{1}{2}(1+\cos 2A)$

$$\therefore I = \tfrac{1}{2}\int_0^{\frac{\pi}{8}} (1-\cos 4\theta)\, d\theta = \tfrac{1}{2}[\theta - \tfrac{1}{4}\sin 4\theta]_0^{\frac{\pi}{8}}$$

$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$ 5

$$= \tfrac{1}{2}[(\tfrac{\pi}{8} - \tfrac{1}{4}\sin\tfrac{\pi}{2}) - (0 - \tfrac{1}{4}\sin 0)] = \tfrac{1}{2}[(\tfrac{\pi}{8} - \tfrac{1}{4}) - (0-0)] = \tfrac{\pi}{16} - \tfrac{1}{8}$$

8 (c) (i)

$$I = \int_1^2 \frac{1}{\sqrt{3+2x-x^2}}\, dx$$

$\int \frac{dx}{\sqrt{(a)^2-(x\pm b)^2}} = \sin^{-1}\left(\frac{x\pm b}{a}\right) + c$ 7

Start by working on the quadratic:

$$3+2x-x^2 = -(x^2-2x-3) = -(x^2-2x+1-4)$$

$$= -((x-1)^2 - 2^2) = (2)^2 - (x-1)^2$$

$$= \left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{0}{2}\right)\right] = \sin^{-1}\tfrac{1}{2} = \tfrac{\pi}{6}$$

8 (c) (ii)

Ex. Use integration methods to derive a formula for the volume of a cone.

Solution

Slope of $L = \frac{r}{h} \Rightarrow$ Equation of L: $rx - hy = 0 \Rightarrow y = \frac{r}{h}x$

Volume of cone $V = \pi \int_0^r y^2\, dx$

$$= \pi \frac{r^2}{h^2}\int_0^h x^2\, dx = \pi \frac{r^2}{3h^2}[x^3]_0^h = \pi \frac{r^2}{3h^2}[h^3 - 0]$$

$$= \tfrac{1}{3}\pi r^2 h$$