INTEGRATION (Q 8, PAPER 1)

2004

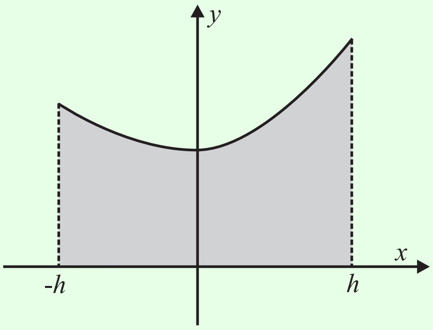
8 (a) Find (i) $\int \frac{1}{x^2}\,dx$ (ii) $\int \cos 6x\,dx$

8 (b) Evaluate (i) $\int_3^6 \frac{dx}{\sqrt{36-x^2}}$ (ii) $\int_0^{\frac{\pi}{3}} \sin x \cos^3 x\,dx$

8 (c) The graph of the function $f(x) = ax^2 + bx + c$ from $x = -h$ to $x = h$ is shown in the diagram.

(i) Show that the area of the shaded region is $\frac{h}{3}[2ah^2 + 6c]$.

(ii) Given that $f(-h) = y_1$, $f(0) = y_2$ and $f(h) = y_3$, express the area of the shaded region in terms of y_1, y_2, y_3 and h.



SOLUTION

8 (a) (i)

$\int x^p\,dx = \frac{x^{p+1}}{p+1} + c$ for $p \in \mathbf{R}$ except $p \neq -1$. **1**

Remember it as:

Add one to the power of *x* and divide by the new power.

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

8 (a) (ii)

$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$ **5**

$$\int \cos 6x\,dx = \tfrac{1}{6}\sin 6x + c$$

8 (b) (i)

$\int \frac{dx}{\sqrt{(a)^2 - (x)^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$ **7**

$$\int_3^6 \frac{dx}{\sqrt{36-x^2}} = \int_3^6 \frac{dx}{\sqrt{(6)^2 - (x)^2}} = [\sin^{-1}\tfrac{x}{6}]_3^6$$

$$= [\sin^{-1} 1 - \sin^{-1}\tfrac{1}{2}] = \tfrac{\pi}{2} - \tfrac{\pi}{6} = \tfrac{\pi}{3}$$

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8 (b) (ii)

$$I = \int_0^{\frac{\pi}{3}} \sin x \cos^3 x \, dx$$

TWO ODD POWERS

STEPS
1. Bracket the **HIGHER** powered trig function.
2. Break off one of the other trig functions and bracket with dx.
3. Let u equal the trig function with the higher power.
4. Integrate by substitution as already explained.

1/2. $I = \int_0^{\frac{\pi}{3}} (\cos x)^3 (\sin x \, dx)$

3. Let $u = \cos x \Rightarrow du = -\sin x \, dx \Rightarrow -du = \sin x \, dx$

Changing limits:

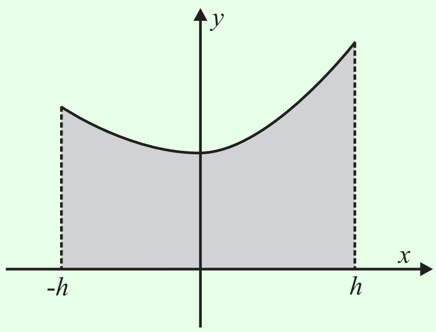
$x = \frac{\pi}{3} \Rightarrow u = \cos\frac{\pi}{3} = \frac{1}{2}$

$x = 0 \Rightarrow u = \cos 0 = 1$

4. $\therefore I = -\int_1^{\frac{1}{2}} u^3 du = -\frac{1}{4}[u^4]_1^{\frac{1}{2}} = -\frac{1}{4}[(\frac{1}{2})^4 - (1)^4] = -\frac{1}{4}[\frac{1}{16} - 1] = -\frac{1}{4}[-\frac{15}{16}] = \frac{15}{64}$

8 (c) (i)

$A = \int_a^b y \, dx$ **8**



$$A = \int_{-h}^{h} (ax^2 + bx + c) \, dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$

$$= \left[\left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch\right) - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - ch\right)\right]$$

$$= \left[\frac{ah^3}{3} + \frac{bh^2}{2} + ch + \frac{ah^3}{3} - \frac{bh^2}{2} + ch\right] = \left[\frac{2ah^3}{3} + 2ch\right] = \frac{h}{3}[2ah^2 + 6c]$$

8 (c) (ii)

$f(-h) = ah^2 - bh + c = y_1$

$f(h) = ah^2 + bh + c = y_3$

$\Rightarrow y_1 + y_3 = 2ah^2 + 2c$

$f(0) = a(0)^2 + b(0) + c = y_2 \Rightarrow c = y_2$

$$A = \frac{h}{3}[2ah^2 + 6c] = \frac{h}{3}[2ah^2 + 2c + 4c] = \frac{h}{3}[y_1 + y_3 + 4y_2]$$