INTEGRATION (Q 8, PAPER 1)

2003

8 (a) Find (i) $\int (x^3+2)\,dx$ (ii) $\int e^{7x}\,dx.$

8 (b) (i) Evaluate $\int_0^1 \frac{2x}{\sqrt{1+x^2}}\,dx$

(ii) By letting $u = \sin x$, evaluate $\int_0^{\frac{\pi}{2}} \cos x \sin^6 x\,dx.$

8 (c) (i) Show that $\int_a^{2a} \sin 2x\,dx = \sin 3a \sin a.$

(ii) Use integration methods to show that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

SOLUTION

8 (a) (i)

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + c \text{ for } p \in \mathbf{R} \text{ except } p \neq -1.$$ 1

Remember it as:

Add one to the power of x and divide by the new power.

$$\int (x^3+2)\,dx = \tfrac{1}{4}x^4 + 2x + c$$

8 (a) (ii)

$$\int e^{ax+b}\,dx = \tfrac{1}{a}e^{ax+b} + c$$ 3

$$\int e^{7x}\,dx = \tfrac{1}{7}e^{7x} + c$$

8 (b) (i)

STEPS
1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

$$I=\int_0^1 \frac{2x}{\sqrt{1+x^2}}\,dx$$

1. Let $u=1+x^2$

2. $\Rightarrow du=2x\,dx$

3. $x=0\Rightarrow u=1+(0)^2=1$ and $x=1\Rightarrow u=1+(1)^2=2$

4. $\therefore I=\int_1^2 \frac{du}{\sqrt{u}}=\int_1^2 u^{-\frac{1}{2}}du$

5. $\Rightarrow I=\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^2=2[\sqrt{u}]_1^2=2[\sqrt{2}-1]$

8 (b) (ii)

$$I=\int_0^{\frac{\pi}{2}} \cos x\sin^6 x\,dx$$

EVEN AND ODD POWER: These are integrals that are products of a trig function with an even power and an odd power. They are done by substitution.

STEPS
1. Bracket the **EVEN** power trig function.
2. Break off one of the odd power trig functions and bracket it with dx.
3. Let u equal the trig function with the even power.
4. Integrate by substitution as already explained.

1/2. $I=\int_0^{\frac{\pi}{2}}(\sin x)^6(\cos x\,dx)$

3. Let $u=\sin x\Rightarrow du=\cos x\,dx$

4. $\therefore I=\int_0^1 u^6du=\frac{1}{7}[u^7]_0^1=\frac{1}{7}[1-0]=\frac{1}{7}$

Changing limits:

$x=\frac{\pi}{2}\Rightarrow u=\sin\frac{\pi}{2}=1$

$x=0\Rightarrow u=\sin 0=0$

8 (c) (i)

$$\int_a^{2a} \sin 2x\,dx = -\tfrac{1}{2}[\cos 2x]_a^{2a} = -\tfrac{1}{2}[\cos 4a - \cos 2a]$$

$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+c$ 4

$$= -\tfrac{1}{2}[-2\sin 3a \sin a] = \sin 3a \sin a$$

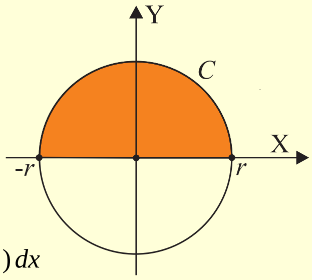
SUMS → PRODUCTS
$\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$
$\sin A - \sin B = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$
$\cos A + \cos B = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$
$\cos A - \cos B = -2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$

8 (c) (ii)

Use integration methods to derive a formula for the volume of a sphere.

SOLUTION

Rotate the circle C with centre (0, 0) and radius r about the X-axis.



Equation of C: $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

Volume of sphere $V = \pi \int_{-r}^{r} y^2\,dx = \pi \int_{-r}^{r} (r^2 - x^2)\,dx$

$= \pi[r^2x - \frac{1}{3}x^3]_{-r}^{r} = \pi[(r^3 - \frac{1}{3}r^3) - (-r^3 + \frac{1}{3}r^3)] = \frac{4}{3}\pi r^3$