INTEGRATION (Q 8, PAPER 1)

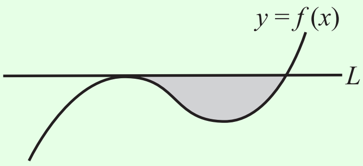
2002

8 (a) Find $\int (x^3 + \sqrt{x} + 2)\, dx.$

8 (b) Evaluate (i) $\int_2^7 \frac{2x-4}{x^2-4x+29}\, dx$ (ii) $\int_2^7 \frac{1}{x^2-4x+29}\, dx.$

8 (c) Let $f(x) = x^3 - 3x^2 + 5.$

L is the tangent to the curve $y = f(x)$ at its local maximum point.
Find the area enclosed between L and the curve.



SOLUTION

8 (a)

$$\int x^p\, dx = \frac{x^{p+1}}{p+1} + c \text{ for } p \in \mathbf{R} \text{ except } p \neq -1. \quad \text{.......} \; \textbf{1}$$

Remember it as:

Add one to the power of x and divide by the new power.

$$\int (x^3 + \sqrt{x} + 2)\, dx = \int (x^3 + x^{\frac{1}{2}} + 2)\, dx = \tfrac{1}{4}x^4 + \tfrac{2}{3}x^{\frac{3}{2}} + 2x + c$$

8 (b)

The two problems in this section are very similar dut done in very different ways. The first is done by sustitution and the second is an inverse quadratic.

8 (b) (i)

$$I = \int_2^7 \frac{2x-4}{x^2-4x+29}\, dx$$

STEPS
1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

1. Let $u = x^2 - 4x + 29$

2. $\Rightarrow du = (2x-4)dx$

3. $x = 2 \Rightarrow u = 2^2 - 4(2) + 29 = 25$ and $x = 7 \Rightarrow u = 7^2 - 4(7) + 29 = 50$

4. $\therefore I = \int_{25}^{50} \frac{du}{u}$

5. $\Rightarrow I = [\ln u]_{25}^{50} = \ln 50 - \ln 25 = \ln \frac{50}{25} = \ln 2$

8 (b) (i)

$$I=\int_2^7 \frac{1}{x^2-4x+29}\,dx$$

$$\int \frac{dx}{(a)^2+(x\pm b)^2}=\frac{1}{a}\tan^{-1}\left(\frac{x\pm b}{a}\right)+c$$ 6

Firstly, sort out the quadratic: $x^2-4x+29=x^2-4x+4+25=(x-2)^2+(5)^2$

$$\therefore I=\int_2^7 \frac{1}{(x-2)^2+(5)^2}\,dx=\left[\frac{1}{5}\tan^{-1}\left(\frac{x-2}{5}\right)\right]_2^7$$

$$=\tfrac{1}{5}[\tan^{-1}1-\tan^{-1}0]=\tfrac{1}{5}[\tfrac{\pi}{4}-0]=\tfrac{\pi}{20}$$

8 (c)

Firstly, find the local maximum point.

$$\text{Turning Point} \Rightarrow \frac{dy}{dx}=0$$ 11

To find out if the turning point (TP) is a local maximum or local minimum:

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}<0$

Local Mininimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}>0$ 12

$$y=x^3-3x^2+5\Rightarrow\frac{dy}{dx}=3x^2-6x$$

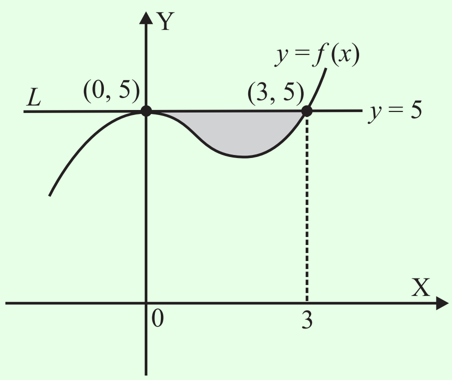
Put $\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=3x^2-6x=0\Rightarrow 3x(x-2)=0\Rightarrow x=0,\ 2$

At $x=0\Rightarrow(0)^3-3(0)^2+5=5\Rightarrow(0,\ 5)$ is a turning point.

$x=2\Rightarrow(2)^3-3(2)^2+5=1\Rightarrow(2,\ 1)$ is a turning point.

Which turning point is the local maximum point?

$$\frac{d^2y}{dx^2}=6x-6\Rightarrow\left(\frac{d^2y}{dx^2}\right)_{x=0}=6(0)-6=-6<0\Rightarrow$$ (0, 5) is the local maximum point.



Now work on the diagram putting the axes in the right place.
As you can see, the line L has equation $y = 5$.
To find the other point of intersection between the line ($y = 5$) and the curve ($y=x^3-3x^2+5$), equate these equations and solve for x.

$$\Rightarrow 5=x^3-3x^2+5\Rightarrow x^3-3x^2=0\Rightarrow x^2(x-3)=0$$

$$\Rightarrow x=0,\ 3$$

The points of intersection are (0, 5) and (3, 5).

Area of shaded part = Area under L − Area under the curve

$$\therefore A = \int_0^3 5\,dx - \int_0^3 (x^3 - 3x^2 + 5)\,dx$$

$$= [5x]_0^3 - [\tfrac{1}{4}x^4 - x^3 + 5x]_0^3 = [15 - 0] - [(\tfrac{1}{4}(3)^4 - (3)^3 + 5(3)) - (0)]$$

$$= 15 - \tfrac{81}{4} + 27 - 15 = \tfrac{27}{4}$$