INTEGRATION (Q 8, PAPER 1)

2001

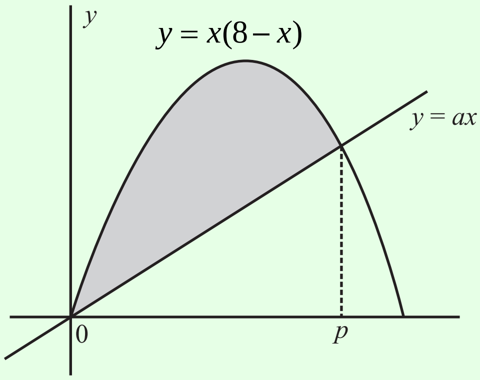
8 (a) Find (i) $\int \frac{1}{x^3}dx$ (ii) $\int \sin 5x\, dx$.

8 (b) Evaluate (i) $\int_0^3 \frac{12}{x^2+9}dx$ (ii) $\int_0^4 \frac{(x+4)}{\sqrt{x^2+8x+1}}dx$.

8 (c) a is a real number such that $0 < a < 8$.
The line $y = ax$ intersects the curve $y = x(8-x)$ at $x = 0$ and at $x = p$.

(i) Show that $p = 8 - a$.

(ii) Show that the area between the curve and the line is $\frac{p^3}{6}$ square units.



SOLUTION

8 (a) (i)

$\int x^p dx = \frac{x^{p+1}}{p+1} + c$ for $p \in \mathbf{R}$ except $p \neq -1$. 1

Remember it as:

Add one to the power of x and divide by the new power.

$$\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$$

8 (a) (ii)

$\int \sin(ax+b)dx = -\tfrac{1}{a}\cos(ax+b) + c$ 4

$$\int \sin 5x dx = -\tfrac{1}{5}\cos 5x + c$$

8 (b) (i)

$\int \frac{dx}{(a)^2+(x)^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ 6

$$I = \int_0^3 \frac{12}{x^2+9}dx = 12\int_0^3 \frac{1}{(x)^2+(3)^2}dx = \tfrac{12}{3}[\tan^{-1}\tfrac{x}{3}]_0^3$$

$$= 4[\tan^{-1}1 - \tan^{-1}0] = 4[\tfrac{\pi}{4} - 0] = \pi$$

8 (b) (ii)

$$I=\int_0^4 \frac{(x+4)}{\sqrt{x^2+8x+1}}dx$$

Steps
1. Let u equal the *inside* of the more complicated function.
2. Differentiate u with respect to x.
3. Change the limits from x to u.
4. Make the substitution.
5. Evaluate the integral I.

1. Let $u=x^2+8x+1$

2. $du=(2x+8)dx=2(x+4)dx \Rightarrow \frac{1}{2}du=(x+4)dx$

3. $x=0\Rightarrow u=(0)^2+8(0)+1=1$ and $x=4\Rightarrow u=(4)^2+8(4)+1=49$

4. $\therefore I=\frac{1}{2}\int_1^{49}\frac{du}{\sqrt{u}}=\frac{1}{2}\int_1^{49}u^{-\frac{1}{2}}du$

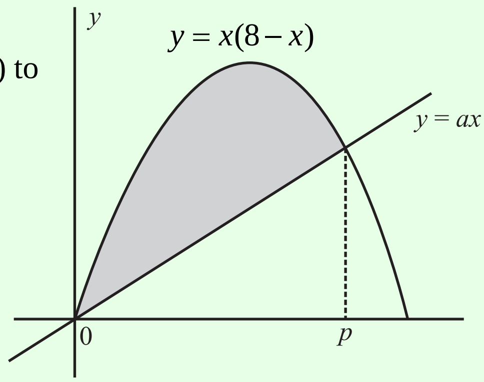
5. $\Rightarrow I=\frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^{49}=[u^{\frac{1}{2}}]_1^{49}=[\sqrt{49}-\sqrt{1}]=[7-1]=6$

8 (c) (i)

Equate the line ($y=ax$) and the curve ($y=x(8-x)$) to find out where they intersect. You are also told they intersect at $x=0$ and $x=p$.

$\Rightarrow ax=x(8-x)\Rightarrow ax=8x-x^2$

At $x=p$: $\Rightarrow ap=8p-p^2\Rightarrow a=8-p\Rightarrow p=8-a$



8 (c) (ii)

Shaded area = Area under the curve − Area under the line

$$\therefore A=\int_0^p(8x-x^2)dx-\int_0^p ax\,dx$$

$$=[4x^2-\tfrac{1}{3}x^3]_0^p-[\tfrac{1}{2}ax^2]_0^p=[(4p^2-\tfrac{1}{3}p^3)-(0)]-[(\tfrac{1}{2}ap^2)-(0)]$$

$$=4p^2-\tfrac{1}{3}p^3-\tfrac{1}{2}ap^2=4p^2-\tfrac{1}{3}p^3-\tfrac{1}{2}(8-p)p^2$$

$$=4p^2-\tfrac{1}{3}p^3-4p^2+\tfrac{1}{2}p^3=\frac{p^3}{6}$$