2011

6. **(a)** Differentiate $\cos^2 x$ with respect to x.

(b) The equation of a curve is $y = e^{-x^2}$.

(i) Find $\frac{dy}{dx}$.

(ii) Find the co-ordinates of the turning point of the curve.

(iii) Determine whether this turning point is a local maximum or a local minimum.

(c) The function f is defined as $x \to \frac{2x}{x+1}$, where $x \in \mathbb{R} \setminus \{-1\}$.

(i) Find the equations of the asymptotes of the curve $y = f(x)$.

(ii) P and Q are distinct points on the curve $y = f(x)$.
The tangent at Q is parallel to the tangent at P.
The co-ordinates of P are $(1, 1)$.
Find the co-ordinates of Q.

(iii) Verify that the point of intersection of the asymptotes is the midpoint of $[PQ]$.

SOLUTION

6 (a)

$y = \cos^2 x = (\cos x)^2$

$\frac{dy}{dx} = 2(\cos x)(-\sin x) = -2\cos x \sin x$

$y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$

$y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$

6 (b) (i)

$y = e^{-x^2}$

$\frac{dy}{dx} = -2xe^{-x^2}$

$y = e^{f(x)} \Rightarrow \frac{dy}{dx} = e^{f(x)} \times f'(x)$

REMEMBER IT AS:

Repeat the whole function × Differentiation of the power.

6 (b) (ii)

$$\frac{dy}{dx} = 0 \Rightarrow -2xe^{-x^2} = 0$$

$$-\frac{2x}{e^{x^2}} = 0$$

$$-2x = 0$$

$$\therefore x = 0$$

$$x = 0 : y = e^{-0} = \frac{1}{e^0} = 1$$

$$\therefore \text{TP: } (0, 1)$$

Turning Point $\Rightarrow \frac{dy}{dx} = 0$

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

6 (b) (iii)

$$\frac{dy}{dx} = -2xe^{-x^2}$$

$$\frac{d^2y}{dx^2} = (-2x)e^{-x^2}(-2x) + e^{-x^2}(-2)$$

$$= 4x^2e^{-x^2} - 2e^{-x^2}$$

$$= e^{-x^2}(4x^2 - 2)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = 4(0)^2 - 2 = -2 < 0 \Rightarrow (0, 1) \text{ is a local maxium.}$$

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}} < 0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}} > 0$

6 (c) (i)

Finding the vertical asymptote: Put the denominator equal to zero.

$$(x+1) = 0 \Rightarrow x = -1$$

Finding the horizontal asymptote: Find $\lim_{x\to\infty} y$.

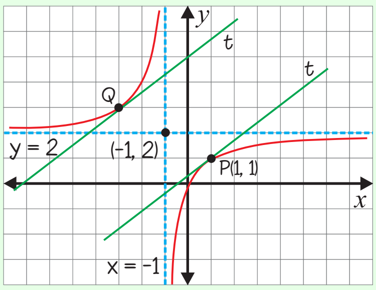
$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(\frac{2x}{x+1}\right)$$

$$= \lim_{x\to\infty}\left(\frac{2\not{x}}{\not{x}(1+\frac{1}{x})}\right)$$

$$= 2$$

$$\therefore y = 2$$

Sketch the situation:



6 (c) (ii)

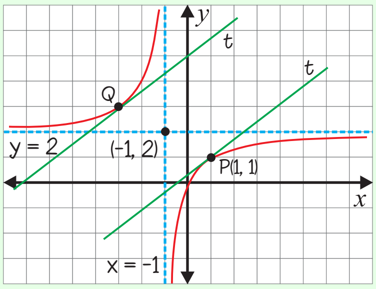
$$f(x)=\frac{2x}{x+1}$$

$$f'(x)=\frac{(x+1)(2)-(2x)(1)}{(x+1)^2}$$ [Differentiate the function to get an expression for slope.]

$$=\frac{2x+2-2x}{(x+1)^2}$$

$$=\frac{2}{(x+1)^2}$$

$$f'(1)=\frac{2}{((1)+1)^2}=\frac{2}{4}=\frac{1}{2}$$ [Find the slope at $x=1$.]



$$f'(x)=\tfrac{1}{2}$$

$$\Rightarrow \frac{2}{(x+1)^2}=\frac{1}{2}$$ [Find the values of x which have this slope. Parallel tangents have the same slope.]

$$4=(x+1)^2$$

$$\pm 2=x+1$$

$$\therefore x=1,\ -3$$

$$x=-3:\ f(-3)=\frac{2(-3)}{(-3)+1}=\frac{-6}{-2}=3$$

$$\therefore Q(-3,\ 3)$$

6 (c) (iii)

$P(1,\ 1),\ Q(-3,\ 3)$

$$\text{Midpoint of } [PQ]=\left(\frac{1-3}{2},\ \frac{1+3}{2}\right)=(-1,\ 2)$$

As you can see from the diagram the midpoint of [PQ] is is equal to the point of intersection of the asymptotes.

7. **(a)** Find the slope of the tangent to the curve $x^2 + y^3 = x - 2$ at the point (3, –2).

(b) A curve is defined by the parametric equations

$$x = \frac{t-1}{t+1} \text{ and } y = \frac{-4t}{(t+1)^2}, \text{ where } t \neq -1.$$

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

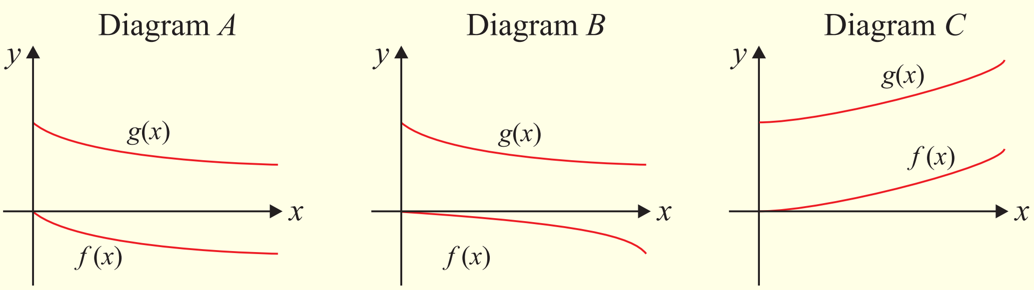
(ii) Hence find $\frac{dy}{dx}$, and express your answer in terms of x.

(c) The functions f and g are defined on the domain $x \in \mathbb{R} \setminus \{-1, 0\}$ as follows:

$$f : x \to \tan^{-1}\left(\frac{-x}{x+1}\right) \text{ and } g : x \to \tan^{-1}\left(\frac{x+1}{x}\right).$$

(i) Show that $f'(x) = \frac{-1}{2x^2 + 2x + 1}$.

(ii) It can be shown that $f'(x) = g'(x)$.
One of the three diagrams *A*, *B*, or *C* below represents parts of the graphs of f and g. Based only on the derivatives, state which diagram is the correct one, and state also why each of the other two diagrams is incorrect.



SOLUTION

7 (a)

$$2x + 3y^2 \times \frac{dy}{dx} = 1$$

$$3y^2 \times \frac{dy}{dx} = 1 - 2x$$

$$\frac{dy}{dx} = \frac{1-2x}{3y^2}$$

$$\left(\frac{dy}{dx}\right)_{(3,\,-2)} = \frac{1-2(3)}{3(-2)^2} = \frac{1-6}{3(4)} = -\frac{5}{12}$$

7 (b) (i)

Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$x = \frac{t-1}{t+1}$$

$$\frac{dx}{dt} = \frac{(t+1)(1)-(t-1)(1)}{(t+1)^2}$$

$$= \frac{t+1-t+1}{(t+1)^2}$$

$$= \frac{2}{(t+1)^2}$$

$$y = \frac{-4t}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{(t+1)^2(-4)-(-4t)2(t+1)(1)}{(t+1)^4}$$

$$= \frac{-4(t^2+2t+1)+8t(t+1)}{(t+1)^4}$$

$$= \frac{-4t^2-8t-4+8t^2+8t}{(t+1)^4}$$

$$= \frac{4t^2-4}{(t+1)^4}$$

$$= \frac{4(t^2-1)}{(t+1)^4} = \frac{4(t-1)(t+1)}{(t+1)^4}$$

$$= \frac{4(t-1)}{(t+1)^3}$$

7 (b) (ii)

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{4(t-1)}{(t+1)^3}}{\frac{2}{(t+1)^2}} = \frac{4(t-1)}{(t+1)^3} \times \frac{(t+1)^2}{2} = \frac{2(t-1)}{(t+1)} = 2x$$

7 (c) (i)

$$f(x) = \tan^{-1}\left(\frac{-x}{x+1}\right)$$

$$f'(x) = \frac{1}{1+\left(\frac{-x}{x+1}\right)^2} \times \frac{(x+1)(-1)-(-x)(1)}{(x+1)^2}$$

$y = \tan^{-1} f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{1+f(x)^2} \times f'(x)$

$$= \frac{1}{1+\frac{x^2}{(x+1)^2}} \times \frac{-x-1+x}{(x+1)^2}$$

$$= \frac{(x+1)^2}{(x+1)^2+x^2} \times \frac{-1}{(x+1)^2}$$

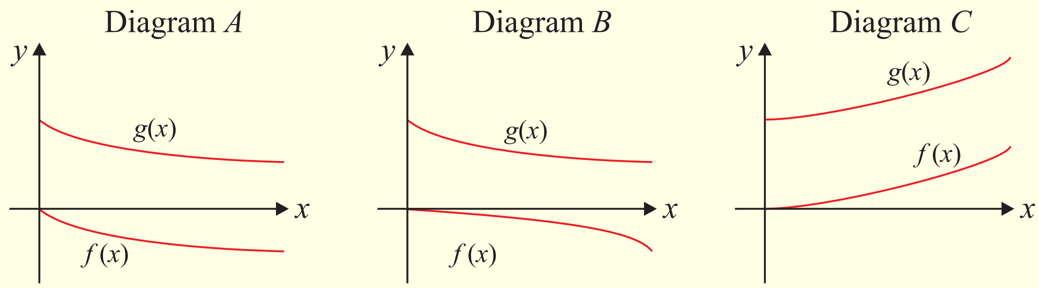
$$= \frac{-1}{x^2+2x+1+x^2}$$

$$= \frac{-1}{2x^2+2x+1}$$

7 (c) (ii)

$$f'(x) = g'(x) = \frac{-1}{2x^2+2x+1} = \frac{-1}{x^2+2x+1+x^2} = \frac{-1}{(x+1)^2+x^2} < 0 \quad \text{for all } x.$$

Therefore, the graph for $f(x)$ is always decreasing.
$g(x)$ has the same slope and is also decreasing.



A is correct: Both functions are decreasing with the same slopes everywhere.
B is incorrect: Both slopes are not the same everywhere.
C is incorrect: Both functions are increasing.