Differentiation & Applications (Q 6 & 7, Paper 1)

2010

6 (a) The equation $x^3 + x^2 - 4 = 0$ has only one real root.

Taking $x_1 = \frac{3}{2}$ as the first approximation to the root, use the Newton-Raphson method to find x_2, the second approximation.

(b) Parametric equations of a curve are:

$$x = \frac{2t-1}{t+2}, \quad y = \frac{t}{t+2}, \text{ where } t \in \mathbf{R} \setminus \{-2\}.$$

(i) Find $\frac{dy}{dx}$.

(ii) What does your answer to part (i) tell you about the shape of the graph?

(c) A curve is defined by the equation $x^2y^3 + 4x + 2y = 12$.

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) Show that the tangent to the curve at the point (0, 6) is also the tangent to it at the point (3, 0).

Solution

6 (a)

$$f(x) = x^3 + x^2 - 4$$
$$f'(x) = 3x^2 + 2x$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$$x_1 = \tfrac{3}{2}$$

$$x_2 = \tfrac{3}{2} - \frac{f(\frac{3}{2})}{f'(\frac{3}{2})} = \tfrac{3}{2} - \frac{(\frac{3}{2})^3 + (\frac{3}{2})^2 - 4}{3(\frac{3}{2})^2 + 2(\frac{3}{2})} = \tfrac{4}{3}$$

6 (b) (i)

$$y=\frac{t}{t+2}$$

$$\frac{dy}{dt}=\frac{(t+2)(1)-t(1)}{(t+2)^2}=\frac{t+2-t}{(t+2)^2}=\frac{2}{(t+2)^2}$$

$$x=\frac{2t-1}{t+2}$$

$$\frac{dx}{dt}=\frac{(t+2)(2)-(2t-1)(1)}{(t+2)^2}=\frac{2t+4-2t+1}{(t+2)^2}=\frac{5}{(t+2)^2}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{2}{(t+2)^2}}{\frac{5}{(t+2)^2}}=\frac{2}{5}$$

6 (b) (ii)

The graph is a straight line.

6 (c) (i)

$$x^2y^3+4x+2y=12$$

$$[x^2\times 3y^2\frac{dy}{dx}+y^3\times 2x]+4+2\frac{dy}{dx}=0$$

$$3x^2y^2\frac{dy}{dx}+2xy^3+4+2\frac{dy}{dx}=0$$

$$(3x^2y^2+2)\frac{dy}{dx}=-2xy^3-4$$

$$\frac{dy}{dx}=\frac{-2xy^3-4}{3x^2y^2+2}$$

6 (c) (ii)

$$\left(\frac{dy}{dx}\right)_{(0,\,6)}=\frac{-2(0)(6)^3-4}{3(0)^2(6)^2+2}=\frac{0-4}{0+2}=\frac{-4}{2}=-2$$

Equation of tangent t_1: Point (0, 6), $m=-2$

$$t_1: 2x+y+k=0$$

$$(0,\,6)\in t_1 \Rightarrow 2(0)+(6)+k=0$$

$$+6+k=0$$

$$k=-6$$

$$t_1: 2x+y-6=0$$

$$\left(\frac{dy}{dx}\right)_{(3,\,0)}=\frac{-2(3)(0)^3-4}{3(3)^2(0)^2+2}=\frac{0-4}{0+2}=\frac{-4}{2}=-2$$

$t_2: 2x+y+k=0$

$(3,\ 0)\in t_2 \Rightarrow 2(3)+(0)+k=0$

$6+k=0$

$k=-6$

$t_2: 2x+y-6=0$

7 (a) Differentiate x^2 with respect to x from first principles.

(b) Let $y=\dfrac{\cos x+\sin x}{\cos x-\sin x}$.

(i) Find $\dfrac{dy}{dx}$.

(ii) Show that $\dfrac{dy}{dx}=1+y^2$.

(c) The function is defined for $x>-1$.

(i) Show that the curve $f(x)=(1+x)\log_e(1+x)$ has a turning point at $\left(\dfrac{1-e}{e},\ -\dfrac{1}{e}\right)$.

(ii) Determine whether the turning point is a local maximum or a local minimum.

Solution

7 (a)

First Principles Proof 2. If $y=x^2 \Rightarrow \dfrac{dy}{dx}=2x$.

Proof

$y+\Delta y=(x+\Delta x)^2=x^2+2x\Delta x+(\Delta x)^2$

$y=x^2$

$\Delta y=2x\Delta x+(\Delta x)^2$ by subtraction

$$\therefore \frac{\Delta y}{\Delta x}=2x+\Delta x \Rightarrow \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=2x$$

7 (b) (i)

$$y=\frac{\cos x+\sin x}{\cos x-\sin x}$$

$$\frac{dy}{dx}=\frac{(\cos x-\sin x)(-\sin x+\cos x)-(\cos x+\sin x)(-\sin x-\cos x)}{(\cos x-\sin x)^2}$$

$$=\frac{-\cos x\sin x+\cos^2 x+\sin^2 x-\sin x\cos x-(-\cos x\sin x-\cos^2 x-\sin^2 x-\sin x\cos x)}{(\cos x-\sin x)^2}$$

$$=\frac{-\cos x\sin x+\cos^2 x+\sin^2 x-\sin x\cos x+\cos x\sin x+\cos^2 x+\sin^2 x+\sin x\cos x}{(\cos x-\sin x)^2}$$

$$=\frac{\cos^2 x+\sin^2 x+\cos^2 x+\sin^2 x}{(\cos x-\sin x)^2}$$

$$=\frac{2}{(\cos x-\sin x)^2}$$

7 (b) (ii)

LHS	**RHS**
$\frac{dy}{dx}$	$1+y^2$
$=\frac{2}{(\cos x-\sin x)^2}$	$=1+\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)^2$
	$=1+\frac{\cos^2+2\cos x\sin x+\sin^2 x}{\cos^2 x-2\cos x\sin x+\sin^2 x}$
	$=\frac{\cos^2 x-2\cos x\sin x+\sin^2 x+\cos^2+2\cos x\sin x+\sin^2 x}{\cos^2 x-2\cos x\sin x+\sin^2 x}$
	$=\frac{\cos^2 x+\sin^2 x+\cos^2+\sin^2 x}{(\cos x-\sin x)^2}$
	$=\frac{2}{(\cos x-\sin x)^2}$

7 (c) (i)

$$f(x)=(1+x)\log_e(1+x)$$

$$f'(x)=(1+x)\times\frac{1}{(1+x)}+[\log_e(1+x)]\times 1$$

$$=1+\log_e(1+x)$$

$$f'(x)=0\Rightarrow 1+\log_e(1+x)=0$$

Turning Point $\Rightarrow \frac{dy}{dx}=0$

$$\log_e(1+x)=-1$$

$$1+x=e^{-1}$$

$$x=\frac{1}{e}-1$$

$$x=\frac{1-e}{e}$$

$$x=\frac{1-e}{e}$$

$$\begin{aligned} f(x) &= \left(1+\frac{1-e}{e}\right)\log_e\left(1+\frac{1-e}{e}\right) \\ &= \left(\frac{e+1-e}{e}\right)\log_e\left(\frac{e+1-e}{e}\right) \\ &= \left(\frac{1}{e}\right)\log_e\left(\frac{1}{e}\right) \\ &= \left(\frac{1}{e}\right)[\log_e 1-\log_e e] \\ &= \left(\frac{1}{e}\right)[0-1]=-\frac{1}{e} \end{aligned}$$

Turning point: $\left(\frac{1-e}{e},\ -\frac{1}{e}\right)$

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}<0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}}>0$

7 (c) (ii)

$$f'(x)=1+\log_e(1+x)$$

$$f''(x)=\frac{1}{1+x}$$

$$f''\left(\frac{1-e}{e}\right)=\frac{1}{1+\left(\frac{1-e}{e}\right)}=\frac{1}{\frac{e+1-e}{e}}=\frac{1}{\frac{1}{e}}=e>0\Rightarrow\left(\frac{1-e}{e},\ -\frac{1}{e}\right)$$ is a local minimum.