DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

2009

6 (a) Differentiate $\sin(3x^2 - x)$ with respect to x.

(b) (i) Differentiate $\sqrt{x}$ with respect to x, from first principles.

(ii) An object moves in a straight line such that its distance from a fixed point is given by $s = \sqrt{t^2 + 1}$, where s is in metres and t is in seconds.
Find the speed of the object when $t = 5$ seconds.

(c) The equation of a curve is $y = \dfrac{2}{x-3}$.

(i) Write down the equations of the asymptotes and hence sketch the curve.

(ii) Prove that no two tangents to the curve are perpendicular to each other.

SOLUTION

6 (a)

$$y = \sin(3x^2 - x)$$
$$\frac{dy}{dx} = [\cos(3x^2 - x)] \times (6x - 1)$$
$$= (6x - 1)\cos(3x^2 - x)$$

$$y = \sin f(x) \Rightarrow \frac{dy}{dx} = \cos f(x) \times f'(x)$$

6 (b) (i)

FIRST PRINCIPLES PROOF 5. If $y = \sqrt{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$.

PROOF

$$y + \Delta y = \sqrt{x + \Delta x}$$
$$y = \sqrt{x}$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = (\sqrt{x + \Delta x} - \sqrt{x}) \times \frac{(\sqrt{x + \Delta x} + \sqrt{x})}{(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$= \frac{x + \Delta x - x}{(\sqrt{x + \Delta x} + \sqrt{x})} = \frac{\Delta x}{(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{(\sqrt{x + \Delta x} + \sqrt{x})} \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}$$

6 (b) (ii)

$$s = \sqrt{t^2+1} = (t^2+1)^{\frac{1}{2}}$$

$$v = \frac{ds}{dt} = \tfrac{1}{2}(t^2+1)^{-\frac{1}{2}}(2t) = \frac{t}{\sqrt{t^2+1}}$$

$$v = \left(\frac{ds}{dt}\right)_{t=5} = \frac{5}{\sqrt{5^2+1}} = \frac{5}{\sqrt{26}} \text{ m/s}$$

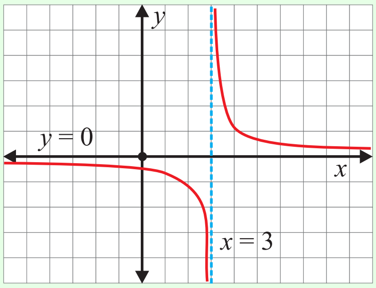
$$v = \frac{ds}{dt}$$

6 (c) (i)

Put the denominator equal to zero to find the vertical asymptote of the curve.
Find $\lim\limits_{x\to\infty} y$ to find the horizontal asymptote of the curve.

Vertical asymptote: $x-3=0 \Rightarrow x=3$

Horizontal asymptote: $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty}\left(\frac{2}{x-3}\right) = 0$



6 (c) (ii)

$$y = \frac{2}{x-3} = 2(x-3)^{-1}$$

$$m = \frac{dy}{dx} = -2(x-2)^{-2} = -\frac{2}{(x-2)^2}$$

The slope m of the tangent to the curve is negative for all values of x. A perpendicular tangent needs a positive slope as the product of perpendicular slopes is equal to –1 ($m_1 \times m_2 = -1$). Therefore, no two tangents are perpendicular as there are no tangents that have positive slopes.

7 (a) The equation of a curve is $x^2 - y^2 = 25$. Find $\frac{dy}{dx}$ in terms of x and y.

(b) A curve is defined by the parametric equations

$$x = \frac{3t}{t^2 - 2} \text{ and } y = \frac{6}{t^2 - 2}, \text{ where } t \neq \pm\sqrt{2}.$$

(i) Find $\frac{dy}{dx}$ in terms of t.

(ii) Find the equation of the tangent to the curve at the point given by $t = 2$.

(c) The function $f(x) = x^3 - 3x^2 + 3x - 4$ has only one root.

(i) Show that the root lies between 2 and 3.

Anne and Barry are each using the Newton-Raphson method to approximate the root. Anne is starting with 2 as a first approximation and Barry is starting with 3.

(ii) Show that Anne's starting approximation is closer to the root than Barry's. (That is, show that the root is less than 2·5.)

(iii) Show, however, that Barry's next approximation is closer to the root than Anne's.

SOLUTION

7 (a)

$$x^2 - y^2 = 25$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$x = y\frac{dy}{dx}$$

$$\frac{x}{y} = \frac{dy}{dx}$$

7 (b) (i)

$$y=\frac{6}{t^2-2}=6(t^2-2)^{-1}$$

$$\frac{dy}{dt}=-6(t^2-2)^{-2}(2t)=-\frac{12t}{(t^2-2)^2}$$

$$x=\frac{3t}{t^2-2}$$

$$\frac{dx}{dt}=\frac{(t^2-2)3-3t(2t)}{(t^2-2)^2}=\frac{3t^2-6-6t^2}{(t^2-2)^2}=\frac{-3t^2-6}{(t^2-2)^2}=-\frac{3t^2+6}{(t^2-2)^2}$$

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-\frac{12t}{(t^2-2)^2}}{-\frac{3t^2+6}{(t^2-2)^2}}=\frac{12t}{3t^2+6}=\frac{4t}{t^2+2}$$

7 (b) (ii)

$$m=\left(\frac{dy}{dx}\right)_{t=2}=\frac{4(2)}{(2)^2+2}=\frac{8}{6}=\frac{4}{3}$$

$$x=\frac{3(2)}{(2)^2-2}=\frac{6}{2}=3$$

$$y=\frac{6}{(2)^2-2}=\frac{6}{2}=3$$

Equation of tangent t: Point (3, 3), $m=\frac{4}{3}$

$$t: 4x-3y+k=0$$

$$(3,\ 3)\in t \Rightarrow 4(3)-3(3)+k=0$$

$$12-9+k=0$$

$$3+k=0$$

$$k=-3$$

$$t: 4x-3y-3=0$$

7 (c) (i)

$$f(x) = x^3 - 3x^2 + 3x - 4$$

$$f(2) = (2)^3 - 3(2)^2 + 3(2) - 4 = 8 - 12 + 6 - 4 = -2 < 0$$

$$f(3) = (3)^3 - 3(3)^2 + 3(3) - 4 = 27 - 27 + 9 - 4 = 5 > 0$$

There is a root between 2 and 3 and the sign of the function changes for $f(2)$ and $f(3)$.

7 (c) (ii)

Anne: $x_1 = 2$
Barry: $x_1 = 3$

$$f(2.5) = (2.5)^3 - 3(2.5)^2 + 3(2.5) - 4 = 0.375 > 0$$

As the $f(2.5)$ is greater than zero, the root lies between 2 and 2.5 which means that Anne's starting approximation is closer to the root.

7 (c) (iii)

$$f(x) = x^3 - 3x^2 + 3x - 4$$

$$f'(x) = 3x^2 - 6x + 3$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Anne: $x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-2}{3(2)^2 - 6(2) + 3}$

$$= 2 + \frac{2}{12 - 12 + 3} = 2 + \frac{2}{3} = \frac{8}{3} \approx 2.67$$

Barry: $x_2 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{5}{3(3)^2 - 6(3) + 3}$

$$= 3 - \frac{5}{27 - 18 + 3} = 3 - \frac{5}{12} = \frac{31}{12} \approx 2.58$$