DIFFERENTIATION & APPLICATIONS (Q 6 & 7, PAPER 1)

2008

6 (a) Differentiate $\sqrt{x^3}$ with respect to x.

(b) Let $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Show that $\frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}$.

(c) The function $f(x) = 2x^3 + 3x^2 + bx + c$ has a local maximum at $x = -2$.

(i) Find the value of b.

(ii) Find the range of values of c for which $f(x) = 0$ has three distinct real roots.

SOLUTION

6 (a)

$y = \sqrt{x^3} = x^{\frac{3}{2}}$

$\therefore \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$ **1**

6 (b) **THE QUOTIENT RULE:** If $y = \frac{u}{v}$ then:

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ **4**

$y = e^x \Rightarrow \frac{dy}{dx} = e^x$ .... **7** Formula 7 can be extended to:

$y = e^{f(x)} \Rightarrow \frac{dy}{dx} = e^{f(x)} \times f'(x)$ **7**

REMEMBER IT AS:

Repeat the whole function × Differentiation of the power.

$u = e^x - e^{-x} \Rightarrow \frac{du}{dx} = e^x + e^{-x}$

$v = e^x + e^{-x} \Rightarrow \frac{dv}{dx} = e^x - e^{-x}$

$u = e^x - e^{-x} \Rightarrow \frac{du}{dx} = e^x + e^{-x}$

$v = e^x + e^{-x} \Rightarrow \frac{dv}{dx} = e^x - e^{-x}$

$\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{2x} + e^0 + e^0 + e^{-2x} - e^{2x} + e^0 + e^0 - e^{-2x}}{(e^x + e^{-x})^2}$ [$e^0 = 1$]

$\therefore \frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}$

6 (c) (i)

$f(x) = 2x^3 + 3x^2 + bx + c$

$\Rightarrow f'(x) = 6x^2 + 6x + b$

$\therefore f'(-2) = 6(-2)^2 + 6(-2) + b = 0$

$\Rightarrow 24 - 12 + b = 0$

$\therefore b = -12$

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

6 (c) (ii)

There will be three distinct roots if the maximum and minimum points are on opposite sides of the x-axis.

You know that there is a local maximum at $x = -2$.
Find the other turning point which is a local minimum.

$f'(x) = 0 \Rightarrow 6x^2 + 6x - 12 = 0$

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow (x+2)(x-1) = 0$

$\therefore x = -2, 1$

Therefore, there is a local minimum at $x = 1$.

Three distinct roots $\Rightarrow f(-2) > 0$ and $f(1) < 0$

$f(-2) > 0 \Rightarrow 2(-2)^3 + 3(-2)^2 - 12(-2) + c > 0$

$\Rightarrow -16 + 12 + 24 + c > 0$

$\Rightarrow c + 20 > 0$

$\Rightarrow c > -20$

$f(1) < 0 \Rightarrow 2(1)^3 + 3(1)^2 - 12(1) + c < 0$

$\Rightarrow 2 + 3 - 12 + c < 0$

$\Rightarrow c - 7 < 0$

$\Rightarrow c < 7$

$\therefore -20 < c < 7$

7 (a) Differentiate $2x+\sin 2x$ with respect to x.

(b) The equation of a curve is $5x^2+5y^2+6xy=16$.

(i) Find $\frac{dy}{dx}$ in terms of x and y.

(ii) $(1,1)$ and $(2,-2)$ are two points on the curve.
Show that the tangents at these points are perpendicular to each other.

(c) Let $y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$.

Find $\frac{dy}{dx}$ and express it in the form $\frac{a}{a+x^b}$, where $a, b\in \mathbf{N}$.

Solution

7 (a)

$y=2x+\sin 2x$

$\Rightarrow \frac{dy}{dx}=2+2\cos 2x$

$y=\sin f(x)\Rightarrow \frac{dy}{dx}=\cos f(x)\times f'(x)$ 5

7 (b) (i)

$5x^2+5y^2+6xy=16$

$\therefore 10x+10y\frac{dy}{dx}+6\left[x\frac{dy}{dx}+y(1)\right]=0$

$\Rightarrow 10x+10y\frac{dy}{dx}+6x\frac{dy}{dx}+6y=0$

$\Rightarrow 5x+5y\frac{dy}{dx}+3x\frac{dy}{dx}+3y=0$

$\Rightarrow (3x+5y)\frac{dy}{dx}=-5x-3y$

$\therefore \frac{dy}{dx}=-\frac{5x+3y}{3x+5y}$

7 (b) (ii)

$\left(\frac{dy}{dx}\right)_{(1,1)}=-\frac{5(1)+3(1)}{3(1)+5(1)}=-\frac{8}{8}=-1$

$\left(\frac{dy}{dx}\right)_{(2,-2)}=-\frac{5(2)+3(-2)}{3(2)+5(-2)}=-\frac{4}{-4}=1$

These two slopes are perpendicular because their product is –1.

7 (c)

$$y=\sin^{-1} f(x) \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-f(x)^2}}\times f'(x)$$ **9**

THE QUOTIENT RULE: If $y=\frac{u}{v}$ then:

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$ **4**

$$y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}}\times\left[\frac{(1+x^2)^{\frac{1}{2}}(1)-x(\frac{1}{2})(1+x^2)^{-\frac{1}{2}}(2x)}{(1+x^2)}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{\frac{1+\cancel{x^2}-\cancel{x^2}}{1+x^2}}}\times\left[\frac{(1+x^2)^{\frac{1}{2}}-x^2(1+x^2)^{-\frac{1}{2}}}{(1+x^2)}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{\frac{1}{1+x^2}}}\times\left[\frac{(1+x^2)^{\frac{1}{2}}-x^2(1+x^2)^{-\frac{1}{2}}}{(1+x^2)}\right]$$

$$\Rightarrow \frac{dy}{dx}=(1+x^2)^{\frac{1}{2}}\left[\frac{(1+x^2)^{\frac{1}{2}}-x^2(1+x^2)^{-\frac{1}{2}}}{(1+x^2)}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left[\frac{(1+x^2)^1-x^2(1+x^2)^0}{(1+x^2)}\right]=\left[\frac{1+x^2-x^2}{(1+x^2)}\right]$$

$$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$

OR Here is a lovely method, very slick.

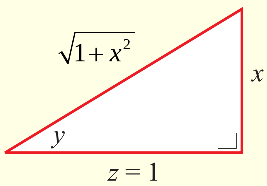
$$y=\sin^{-1}\frac{x}{\sqrt{1+x^2}} \Rightarrow \sin y=\frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \tan y = x$$

$$\Rightarrow y=\tan^{-1}x$$

$$y=\tan^{-1}x \Rightarrow \frac{dy}{dx}=\frac{1}{1+x^2}$$ **10**

$$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$



$$z^2+x^2=(\sqrt{1+x^2})^2$$

$$\Rightarrow z^2+x^2=1+x^2$$

$$\Rightarrow z^2=1$$

$$\therefore z=1$$