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6 (a) Differentiate with respect to x

(i) $(1+5x)^3$ (ii) $\frac{7x}{x-3}, x \neq 3.$

(b) (i) Prove, from first principles, the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $u = u(x)$ and $v = v(x)$.

(ii) Given $y = \sin^{-1}(2x-1)$, find $\frac{dy}{dx}$ and calculate its value at $x = \frac{1}{2}$.

(c) $f(x) = \frac{1}{x+1}$ where $x \in \mathbf{R}, x \neq -1$.

(i) Find the equations of the asymptotes of the graph of $f(x)$.
(ii) Prove that the graph of $f(x)$ has no turning points or points of inflection.
(iii) If the tangents to the curve at $x = x_1$ and $x = x_2$ are parallel and if $x_1 \neq x_2$, show that

$$x_1 + x_2 + 2 = 0.$$

SOLUTION

5 (a) (i)

$y = (1+5x)^3$

$y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$ 1

$\Rightarrow \frac{dy}{dx} = 3(1+5x)^2(5)$

$\therefore \frac{dy}{dx} = 15(1+5x)^2$

5 (a) (ii)

$y = \frac{7x}{x-3}$

$u = 7x \Rightarrow \frac{du}{dx} = 7$

$v = (x-3) \Rightarrow \frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ 4

$\therefore \frac{dy}{dx} = \frac{(x-3)7 - 7x(1)}{(x-3)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{7x - 21 - 7x}{(x-3)^2}$

$\therefore \frac{dy}{dx} = -\frac{21}{(x-3)^2}$

6 (b) (i)

Statement of Sum Rule: If $y = u + v$ then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Proof

y + Dy = (u + Du) + (v + Dv)

y = $u + v$

Dy = Du + Dv

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

6 (b) (ii)

$$y = \sin^{-1} f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - f(x)^2}} \times f'(x)$$ 9

$$y = \sin^{-1}(2x - 1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - (2x-1)^2}} \times 2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{1}{2}} = \frac{2}{\sqrt{1 - (2(\frac{1}{2}) - 1)^2}} = \frac{2}{\sqrt{1 - (1-1)^2}} = \frac{2}{\sqrt{1 - 0}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{1}{2}} = 2$$

6 (c) (i)

Finding the vertical asymptote: Put the denominator equal to zero.

$x + 1 = 0 \Rightarrow x = -1$ is the vertical asymptote.

Finding the horizontal asymptote: Find $\lim\limits_{x \to \infty} y$.

$$y = f(x) = \frac{1}{x+1}$$

$$\Rightarrow \lim_{x \to \infty} y = \lim_{x \to \infty} \frac{1}{x+1} = 0$$

Therefore, $y = 0$ is the horizontal asymptote.

6 (c) (ii)

$$f(x) = y = \frac{1}{x+1} = (x+1)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = -1(1+x)^{-2}(1) = -(1+x)^{-2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2(1+x)^{-3}(1) = 2(1+x)^{-3}$$

$\frac{dy}{dx} = 0 \Rightarrow -\frac{1}{(1+x)^2} = 0$ [Multiply both sides by $(1 + x)^2$.]

$\Rightarrow 1 = 0$ [This is nonsense.]

Therefore, there are no turning points.

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{2}{(1+x)^3} = 0$ [Multiply both sides by $(1 + x)^3$.]

$\Rightarrow 2 = 0$ [This is nonsense.]

Therefore, there are no points of inflection.

To find the point of inflection set $\frac{d^2y}{dx^2} = 0$ and solve for x.

6 (c) (iii)

$\frac{dy}{dx}$ is the slope. The slopes at x_1 and x_2 are the same as their tangents are parallel.

$$\left(\frac{dy}{dx}\right)_{x=x_1} = -\frac{1}{(1+x_1)^2}$$

$$\left(\frac{dy}{dx}\right)_{x=x_2} = -\frac{1}{(1+x_2)^2}$$

$$\therefore -\frac{1}{(1+x_1)^2} = -\frac{1}{(1+x_2)^2}$$

$\Rightarrow (1+x_1)^2 = (1+x_2)^2$

$\Rightarrow 1+x_1 = \pm(1+x_2)$ [Solve each equation separately.]

$1+x_1 = +(1+x_2)$

$\Rightarrow x_1 = x_2$

[Ignore this solution as you are told that $x_1 \neq x_2$.]

$1+x_1 = -(1+x_2)$

$\Rightarrow 1+x_1 = -1-x_2$

$\therefore x_1 + x_2 + 2 = 0$

7 (a) Find the slope of the tangent to the curve $x^2 - xy + y^2 = 1$ at the point (1, 0).

(b) The parametric equations of a curve are $x = \cos^3 t$ and $y = \sin^3 t$, $0 \le t \le \frac{\pi}{2}$.

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

(ii) Hence, find integers a and b such that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{a}{b}(\sin 2t)^2$.

(c) $f(x) = \frac{\ln x}{x}$ where $x > 0$.

(i) Show that the maximum of $f(x)$ occurs at the point $(e, \frac{1}{e})$.

(ii) Hence, show that $x^e \le e^x$ for all $x > 0$.

SOLUTION

7 (a)

$x^2 - xy + y^2 = 1$

$\Rightarrow 2x - [x\frac{dy}{dx} + y(1)] + 2y\frac{dy}{dx} = 0$ [The product rule is used on xy.]

$\Rightarrow 2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$

$\Rightarrow (2y - x)\frac{dy}{dx} = y - 2x$

$\Rightarrow \frac{dy}{dx} = \frac{y - 2x}{2y - x}$

$\therefore \left(\frac{dy}{dx}\right)_{(1,\,0)} = \frac{(0) - 2(1)}{2(0) - (1)} = \frac{-2}{-1} = 2$

7 (b) (i)

$x = \cos^3 t = (\cos t)^3$

$\Rightarrow \frac{dx}{dt} = 3(\cos t)^2(-\sin t)$

$\therefore \frac{dx}{dt} = -3\cos^2 t \sin t$

$y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$ 1

$y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$ 6

$y = \sin^3 t = (\sin t)^3$

$\Rightarrow \frac{dy}{dt} = 3(\sin t)^2(\cos t)$

$\therefore \frac{dy}{dt} = 3\sin^2 t \cos t$

$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$ 5

7 (b) (ii)

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$$

$$=(-3\cos^2\sin t)^2+(3\sin^2 t\cos t)^2$$

$$=9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t$$

$$=9\cos^2 t\sin^2 t(\cos^2 t+\sin^2 t)$$ $\boxed{\cos^2 A+\sin^2 A=1}$ **8**

$$=9\cos^2 t\sin^2 t$$

$$=(3\cos t\sin t)^2$$ $\boxed{\sin 2A=2\sin A\cos A}$ **13**

$$=(\tfrac{3}{2}\times 2\sin t\cos t)^2$$

$$=\tfrac{9}{4}(\sin 2t)^2$$

$$\therefore a=9,\ b=4$$

7 (c) (i)

$$f(x)=y=\frac{\ln x}{x}$$

$$u=\ln x\Rightarrow\frac{du}{dx}=\frac{1}{x}$$

$$v=x\Rightarrow\frac{dv}{dx}=1$$

$$\frac{dy}{dx}=\frac{x(\frac{1}{x})-\ln x}{x^2}$$

$$\therefore\frac{dy}{dx}=\frac{1-\ln x}{x^2}$$

$\boxed{\dfrac{dy}{dx}=\dfrac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}}$ **4**

$$\frac{dy}{dx}=0\Rightarrow\frac{1-\ln x}{x^2}=0$$

To find the turning points set $\frac{dy}{dx}=0$ and solve for x.

$$\Rightarrow 1-\ln x=0$$

$$\Rightarrow 1=\ln x$$

$$\Rightarrow 1=\log_e x$$

$$\Rightarrow e^1=x$$

$$\therefore x=e$$

$$\therefore f(e)=\frac{\ln e}{e}=\frac{1}{e}$$

$\therefore(e,\ \frac{1}{e})$ is a turning point.

You need to show this point is a maximum.

Local Maximum: $\left(\dfrac{d^2y}{dx^2}\right)_{TP}<0$

Local Minimum: $\left(\dfrac{d^2y}{dx^2}\right)_{TP}>0$ **12**

$$\frac{dy}{dx}=\frac{1-\ln x}{x^2}$$

$$u=1-\ln x\Rightarrow\frac{du}{dx}=-\tfrac{1}{x}$$

$$v=x^2\Rightarrow\frac{dv}{dx}=2x$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{x^2(-\frac{1}{x})-(1-\ln x)}{x^4}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x-1+\ln x}{x^4}$$

$$\therefore\left(\frac{d^2y}{dx^2}\right)_{x=e}=\frac{-e-1+\ln e}{e^4}=\frac{-e-1+1}{e^4}=\frac{-e}{e^4}=-\frac{1}{e^3}<0\Rightarrow(e,\ \tfrac{1}{e})\text{ is a local maximum.}$$

7 (c) (ii)

$x^e \le e^x$ [Take the natural log of both sides.]

$\Rightarrow \ln x^e \le \ln e^x$ [When ln and e come together they cancel.]

$\Rightarrow e \ln x \le x$

$\Rightarrow \dfrac{\ln x}{x} \le \dfrac{1}{e}$

But $f(x) = \dfrac{\ln x}{x} \Rightarrow f(x) \le \dfrac{1}{e}$

This is true as $(e, \frac{1}{e})$ is the only maximum point.