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6 (a) Differentiate

$(3-4x)^5$ with respect to x.

(b) Find from first principles the derivative of $\sin x$ with respect to x.

(c) Let $f(x) = xe^{-ax}$, $x \in \mathbf{R}$, a constant and $a > 0$.
Show that $f(x)$ has a local maximum and express the coordinates of this local maximum point in terms of a.
Find, in terms of a, the coordinates of the point at which the second derivative of $f(x)$ is zero.

SOLUTION

6 (a)

$y = (3-4x)^5$

$\Rightarrow \dfrac{dy}{dx} = 5(3-4x)^4(-4)$

$\therefore \dfrac{dy}{dx} = -20(3-4x)^4$

$y = [f(x)]^n \Rightarrow \dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$ 1

6 (b)

FIRST PRINCIPLES PROOF. If $y = \sin x \Rightarrow \dfrac{dy}{dx} = \cos x$.

PROOF

$y + \Delta y = \sin(x + \Delta x)$

$y = \sin x$

$\Delta y = \sin(x+\Delta x) - \sin x = 2\cos(\frac{2x+\Delta x}{2})\sin(\frac{\Delta x}{2})$ by subtraction

$\therefore \dfrac{\Delta y}{\Delta x} = \dfrac{2\cos(\frac{2x+\Delta x}{2})\sin(\frac{\Delta x}{2})}{\Delta x} = \cos(\frac{2x+\Delta x}{2}) \times \dfrac{\sin(\frac{\Delta x}{2})}{(\frac{\Delta x}{2})}$ [**Note**: $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$]

$\Rightarrow \dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \cos x$

6 (c)

$y = f(x) = xe^{-ax}$

$\Rightarrow \frac{dy}{dx} = x(-ae^{-ax}) + e^{-ax}(1)$

$\Rightarrow \frac{dy}{dx} = -axe^{-ax} + e^{-ax}$

$\therefore \frac{dy}{dx} = e^{-ax}(1-ax)$

$\frac{dy}{dx} = 0 \Rightarrow e^{-ax}(1-ax) = 0$

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

$y = e^{f(x)} \Rightarrow \frac{dy}{dx} = e^{f(x)} \times f'(x)$ 7

$u = x \quad \Rightarrow \frac{du}{dx} = 1$

$v = e^{-ax} \Rightarrow \frac{dv}{dx} = -ae^{-ax}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ 3

$e^{-ax} = 0$

$\Rightarrow \ln e^{-ax} = \ln 0$

This has no solutions as the ln 0 does not exist.

$1 - ax = 0$

$\therefore x = \frac{1}{a}$

$y = f(\frac{1}{a}) = (\frac{1}{a})e^{-a(\frac{1}{a})}$

$\Rightarrow y = \frac{1}{a}e^{-1}$

$\therefore y = \frac{1}{ae}$

$\therefore (\frac{1}{a}, \frac{1}{ae})$ is the only turning point.

You need to find out if it is a maximum or minimum.

$\frac{dy}{dx} = e^{-ax}(1-ax)$

$\Rightarrow \frac{d^2y}{dx^2} = e^{-ax}(-a) + (1-ax)(-ae^{-ax})$

$\Rightarrow \frac{d^2y}{dx^2} = -ae^{-ax} - ae^{-ax} + a^2xe^{-ax}$

$\therefore \frac{d^2y}{dx^2} = -2ae^{-ax} + a^2xe^{-ax} = ae^{-ax}(ax-2)$

$u = e^{-ax} \quad \Rightarrow \frac{du}{dx} = -ae^{-ax}$

$v = (1-ax) \Rightarrow \frac{dv}{dx} = -a$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{a}} = ae^{-a(\frac{1}{a})}(a(\frac{1}{a})-2) = ae^{-1}(1-2) = -\frac{a}{e} < 0$

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}} < 0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}} > 0$ 12

$\therefore (\frac{1}{a}, \frac{1}{ae})$ is a local maximum.

$\therefore \frac{d^2y}{dx^2} = 0 \Rightarrow ae^{-ax}(ax-2) = 0$

$\Rightarrow ax - 2 = 0$

$\therefore x = \frac{2}{a}$

$y = f(\frac{2}{a}) = (\frac{2}{a})e^{-a(\frac{2}{a})} = (\frac{2}{a})e^{-2} = \frac{2}{ae^2}$

$\therefore \left(\frac{2}{a}, \frac{2}{ae^2}\right)$ is the solution.

7 (a) Find the derivative of $\sqrt{x^2+1}$.

(b) (i) Let $x = t - \sin t \cos t$ and $y = 4\cos t,\ 0 < t < \frac{\pi}{2}$.

Show that $\dfrac{dy}{dx} = -\dfrac{2}{\sin t}$.

(ii) Find the slope of the tangent to the curve

$x^2 - y^2 - x = 1$ at the point (2, 1).

(c) Let $f(x) = x^3 + kx^2 - 4,\ x \in \mathbf{R}$ and $k > 0$.

Show that the coordinates of the local minimum and local maximum of $f(x)$ are

$(0, -4)$ and $\left(-\dfrac{2k}{3}, \dfrac{4k^3 - 108}{27}\right)$, respectively.

Find

(i) the range of values of k for which $f(x) = 0$ has three real roots

(ii) the value of k for which $f(x) = 0$ has three roots, two of which are equal.

SOLUTION

7 (a)

$$y = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$$

$y = [f(x)]^n \Rightarrow \dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$

.... 1

$$\therefore \frac{dy}{dx} = \frac{x}{(x^2+1)^{\frac{1}{2}}} = \frac{x}{\sqrt{x^2+1}}$$

7 (b) (i)

Do $\dfrac{dy}{dt}$ first, then do $\dfrac{dx}{dt}$, and then divide $\dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \dfrac{dy}{dx}$

$$y = 4\cos t \Rightarrow \frac{dy}{dt} = -4\sin t$$

$y = \cos x \Rightarrow \dfrac{dy}{dx} = -\sin x$

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$$x = t - \sin t \cos t = t - \tfrac{1}{2}\sin 2t$$

$$\Rightarrow \frac{dx}{dt} = 1 - \tfrac{1}{2}[2\cos 2t] = 1 - \cos 2t$$

$y = \cos f(x) \Rightarrow \dfrac{dy}{dx} = -\sin f(x) \times f'(x)$

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$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-4\sin t}{1 - \cos 2t} = \frac{-4\sin t}{1 - (\cos^2 t - \sin^2 t)}$$

$\cos 2A = \cos^2 A - \sin^2 A$

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$$\Rightarrow \frac{dy}{dx} = \frac{-4\sin t}{1 - \cos^2 t + \sin^2 t} = \frac{-4\sin t}{2\sin^2 t} = -\frac{2}{\sin t}$$

$\cos^2 A + \sin^2 A = 1$

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7 (b) (ii)

$$x^2 - y^2 - x = 1$$

$$\Rightarrow 2x - 2y\frac{dy}{dx} - 1 = 0$$

$$\Rightarrow 2x - 1 = 2y\frac{dy}{dx}$$

$$\therefore \frac{2x-1}{2y} = \frac{dy}{dx}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(2,\,1)} = \frac{2(2)-1}{2(1)} = \frac{4-1}{2} = \frac{3}{2}$$

7 (c)

$$y = f(x) = x^3 + kx^2 - 4$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + 2kx$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6x + 2k$$

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 2kx = 0$$

$$x(3x + 2k) = 0$$

$$\therefore x = 0,\ -\tfrac{2k}{3}$$

Turning Point $\Rightarrow \frac{dy}{dx} = 0$ **11**

$x = 0:\ f(0) = (0)^3 + k(0)^2 - 4 = -4 \Rightarrow (0,\,-4)$ is a turning point.

$$x = -\tfrac{2k}{3}:\ f(-\tfrac{2k}{3}) = (-\tfrac{2k}{3})^3 + k(-\tfrac{2k}{3})^2 - 4$$

$$= -\frac{8k^3}{27} + \frac{4k^3}{9} - 4 = \frac{-8k^3 + 12k^3 - 108}{27} = \frac{4k^3 - 108}{27}$$

$\Rightarrow \left(-\frac{2k}{3},\ \frac{4k^3 - 108}{27}\right)$ is a turning point.

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}} < 0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{\text{TP}} > 0$ **12**

$\left(\frac{d^2y}{dx^2}\right)_{x=0} = 6(0) + 2k = 2k > 0 \Rightarrow (0,\,-4)$ is a local minimum.

$\left(\frac{d^2y}{dx^2}\right)_{x=-\frac{2k}{3}} = 6(-\tfrac{2k}{3}) + 2k = -4k + 2k = -2k < 0 \Rightarrow \left(-\frac{2k}{3},\ \frac{4k^3 - 108}{27}\right)$ is a local maximum.

7 (c) (i)

In order for 3 real roots to exist, the local maximum and minimum must be on opposite sides of the *x*-axis. This allows the curve to cut the *x*-axis 3 times.

The local minimum (0, –4) is below the *x*-axis. This means the local maximum must be above the *x*-axis.

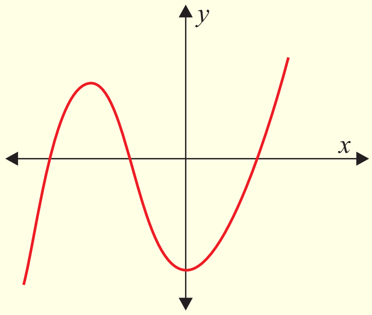
$\therefore \frac{4k^3 - 108}{27} > 0$

$\Rightarrow 4k^3 - 108 > 0$

$\Rightarrow 4k^3 > 108$

$\Rightarrow k^3 > 27$

$\therefore k > 3$



7 (c) (ii)

For 2 roots to be equal, the local maximum must be on the *x*-axis.

$\therefore \frac{4k^3 - 108}{27} = 0$

$\Rightarrow 4k^3 - 108 = 0$

$\Rightarrow 4k^3 = 108$

$\Rightarrow k^3 = 27$

$\therefore k = 3$