1998

6 (a) Differentiate (i) $(1+3x)^2$ (ii) $3e^{4x+1}$.

(b) Find the value of the constant k if $y = kx^2$ is a solution of the equation

$$x\frac{dy}{dx} + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + y = 0,$$

where $x \in \mathbf{R}$ and $k \neq 0$.

(c) Given that $f(x) = \dfrac{x}{x+2}, x \in \mathbf{R}$ and $x \neq 2$,

find the equations of the asymptotes of the graph of $f(x)$.
Prove that the graph of $f(x)$ has no turning points or points of inflection.
Find the range of values of x for which $f'(x) \leq 1$, where $f'(x)$ is the derivative of $f(x)$.

SOLUTION

6 (a) (i)

$y = (1+3x)^2$

$\Rightarrow \dfrac{dy}{dx} = 2(1+3x)^1(3)$

$\therefore \dfrac{dy}{dx} = 6(1+3x)$

$\boxed{y = [f(x)]^n \Rightarrow \dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)}$ 1

6 (a) (ii)

$\boxed{y = e^{f(x)} \Rightarrow \dfrac{dy}{dx} = e^{f(x)} \times f'(x)}$ 7

Repeat the whole function × Differentiation of the power.

$y = 3e^{4x+1}$

$\Rightarrow \dfrac{dy}{dx} = 3[e^{4x+1} \times 4]$

$\therefore \dfrac{dy}{dx} = 12e^{4x+1}$

6 (b)

$y = kx^2$

$\Rightarrow \dfrac{dy}{dx} = 2kx$

$\boxed{y = x^n \Rightarrow \dfrac{dy}{dx} = nx^{n-1}}$ 1

$\therefore x\dfrac{dy}{dx} + \dfrac{1}{2}\left(\dfrac{dy}{dx}\right)^2 + y = 0$

$\Rightarrow x(2kx) + \frac{1}{2}(4k^2x^2) + kx^2 = 0$

$\Rightarrow 2kx^2 + 2k^2x^2 + kx^2 = 0$

$\Rightarrow 3kx^2 + 2k^2x^2 = 0 \Rightarrow kx^2(3+2k) = 0$

$\therefore k = \cancel{0}, -\frac{3}{2}$

6 (c)

Finding the vertical asymptote: Put the denominator equal to zero.

$x+2=0 \Rightarrow x=-2$ is the vertical asymptote.

Finding the horizontal asymptote: Find $\lim\limits_{x\to\infty} y$.

$$f(x)=y=\frac{x}{x+2}$$

$$\lim_{x\to\infty} y=\lim_{x\to\infty}\frac{x}{x+2}=\lim_{x\to\infty}\frac{\cancel{x}}{\cancel{x}(1+\frac{2}{x})}=1$$

Therefore, $y = 1$ is the horizontal asymptote.

$$y=\frac{x}{x+2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(x+2)1-x(1)}{(x+2)^2}=\frac{x+2-x}{(x+2)^2}$$

$$\therefore \frac{dy}{dx}=\frac{2}{(x+2)^2}$$

$$\frac{dy}{dx}=0 \Rightarrow \frac{2}{(x+2)^2}=0$$

$\Rightarrow 2=0$ [This is nonsense.]

Therefore, there are no turning points.

$$u=x \Rightarrow \frac{du}{dx}=1$$

$$v=x+2 \Rightarrow \frac{dv}{dx}=1$$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$ **4**

To find the turning points set $\frac{dy}{dx}=0$ and solve for x.

$$\frac{dy}{dx}=\frac{2}{(x+2)^2}=2(x+2)^{-2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=2[-2(x+2)^{-3}(1)]=-\frac{4}{(x+2)^3}$$

$$\frac{d^2y}{dx^2}=0 \Rightarrow -\frac{4}{(x+2)^3}=0$$

$\Rightarrow -4=0$ [This is nonsense.]

Therefore, there are no points of inflection.

To find the point of inflection set $\frac{d^2y}{dx^2}=0$ and solve for x.

$$f'(x)\le 1 \Rightarrow \frac{2}{(x+2)^2}\le 1$$

$$\Rightarrow 2\le (x+2)^2$$

$$\Rightarrow (x+2)^2\ge 2$$

Solve $(x+2)^2=2$.

$$\Rightarrow (x+2)=\pm\sqrt{2}$$

$$\therefore x=\sqrt{2}-2,\ -\sqrt{2}-2$$

$\boxed{(x+2)^2\ge 2}$ .....**Test Box**

$\leftarrow$ -4 | $\boxed{-\sqrt{2}-2}$ ≈ -3.4 | $\leftrightarrow$ -1 | $\boxed{\sqrt{2}-2}$ ≈ -0.6 | $\rightarrow$ 0

$(-4+2)^2\ge 2$	$(-1+2)^2\ge 2$	$(0+2)^2\ge 2$
$\Rightarrow (-2)^2\ge 2$	$\Rightarrow (1)^2\ge 2$	$\Rightarrow (2)^2\ge 2$
$\Rightarrow 4\ge 2$	$\Rightarrow 1\ge 2$	$\Rightarrow 4\ge 2$
Correct	Wrong	Correct

Answer: $x\le -2-\sqrt{2},\ x\ge -2+\sqrt{2}$

7 (a) Let $\theta = 5t^3 - 2t^2$,

where t is in seconds and θ is in radians.

Find the rate of change of θ when $t = 2$ seconds.

(b) The parametric equations of a curve are

$$x = \frac{1+\sin t}{\cos t},\ y = \frac{1+\cos t}{\sin t},\ 0 < t < \tfrac{\pi}{2}.$$

Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

Find the slope of the tangent to the curve at the point where $t = \tan^{-1}(\tfrac{3}{4})$.

(c) Let $f(x) = x^3 - kx^2 + 8,\ k \in \mathbf{R}$ and $k > 0$.

Show that the coordinates of the local minimum point of $f(x)$ are $(\tfrac{2k}{3}, 8 - \tfrac{4k^3}{27})$.

Taking $x_1 = 3$ as the first approximation of one of the roots of $f(x) = 0$, the Newton-Raphson method gives the second approximation as $x_2 = \tfrac{10}{3}$.

Find the value of k.

Solution

7 (a)

$$\theta = 5t^3 - 2t^2$$

$$\Rightarrow \frac{d\theta}{dt} = 15t^2 - 4t$$

$$\therefore \left(\frac{d\theta}{dt}\right)_{t=2} = 15(2)^2 - 4(2) = 60 - 8 = 52 \text{ rads/s}$$

7 (b)

$$x = \frac{1+\sin t}{\cos t}$$

$$\boxed{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}} \ \ldots\ldots\ 4$$

$$\boxed{\begin{array}{l} u = 1+\sin t \Rightarrow \dfrac{du}{dx} = \cos t \\ v = \cos t \Rightarrow \dfrac{dv}{dx} = -\sin t \end{array}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\cos t(\cos t) - (1+\sin t)(-\sin t)}{(\cos t)^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\cos^2 t + \sin t + \sin^2 t}{(\cos t)^2}$$

$$\therefore \frac{dx}{dt} = \frac{1+\sin t}{\cos^2 t}$$

$$y = \frac{1+\cos t}{\sin t}$$

$$\boxed{y = \sin x \Rightarrow \frac{dy}{dx} = \cos x} \ \ldots\ldots\ 5 \qquad \boxed{y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x} \ \ldots\ldots\ 6$$

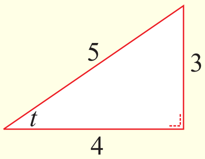
$$\Rightarrow \frac{dy}{dt} = \frac{\sin t(-\sin t) - (1+\cos t)\cos t}{(\sin t)^2}$$

$$\boxed{\begin{array}{l} u = 1+\cos t \Rightarrow \dfrac{du}{dx} = -\sin t \\ v = \sin t \Rightarrow \dfrac{dv}{dx} = \cos t \end{array}}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-\sin^2 t - \cos t - \cos^2 t}{(\sin t)^2} = \frac{-(\sin^2 t + \cos t + \cos^2 t)}{\sin^2 t}$$

$$\therefore \frac{dy}{dt} = \frac{-(\cos t + 1)}{\sin^2 t} = \frac{-1-\cos t}{\sin^2 t}$$

$t = \tan^{-1}(\frac{3}{4}) \Rightarrow \tan t = \frac{3}{4}$



$\therefore \cos t = \frac{4}{5}$

$\therefore \sin t = \frac{3}{5}$

$$\Rightarrow \frac{dx}{dt} = \frac{1+\sin t}{\cos^2 t} = \frac{1+\frac{3}{5}}{(\frac{4}{5})^2} = \frac{\frac{8}{5}}{\frac{16}{25}} = \frac{5}{2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-1-\cos t}{\sin^2 t} = \frac{-1-(\frac{4}{5})}{(\frac{3}{5})^2} = \frac{-1-\frac{4}{5}}{\frac{9}{25}} = \frac{-\frac{9}{5}}{\frac{9}{25}} = -5$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-5}{\frac{5}{2}} = -2$$

7 (c)

$$y = f(x) = x^3 - kx^2 + 8$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 2kx$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6x - 2k$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 2kx = 0$$

$$\Rightarrow x(3x-2k) = 0$$

$$\therefore x = 0,\ \frac{2k}{3}$$

To find the turning points set $\frac{dy}{dx} = 0$ and solve for x.

$$y = f(\tfrac{2k}{3}) = (\tfrac{2k}{3})^3 - k(\tfrac{2k}{3})^2 + 8$$

$$\Rightarrow y = \frac{8k^3}{27} - \frac{4k^3}{9} + 8 = \frac{8k^3 - 12k^3}{27} + 8$$

$$\therefore y = 8 - \frac{4k^3}{27}$$

$\Rightarrow \left(\frac{2k}{3}, 8 - \frac{4k^3}{27}\right)$ is a turning point.

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{2k}{3}} = 6(\tfrac{2k}{3}) - 2k = 4k - 2k = 2k > 0$$

$\Rightarrow \left(\frac{2k}{3}, 8 - \frac{4k^3}{27}\right)$ is a local minimum.

Local Maximum: $\left(\frac{d^2y}{dx^2}\right)_{TP} < 0$

Local Minimum: $\left(\frac{d^2y}{dx^2}\right)_{TP} > 0$

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$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad \ldots\ldots\ 16$$

$$\Rightarrow \frac{10}{3} = 3 - \frac{(3)^3 - k(3)^2 + 8}{3(3)^2 - 2k(3)}$$

$$\Rightarrow \frac{10}{3} = 3 - \frac{27 - 9k + 8}{27 - 6k}$$

$$\Rightarrow \frac{35 - 9k}{27 - 6k} = -\frac{1}{3}$$

$$\Rightarrow 3(35 - 9k) = 6k - 27$$

$$\Rightarrow 35 - 9k = 2k - 9$$

$$\Rightarrow 44 = 11k$$

$$\therefore k = 4$$