1997

6 (a) Differentiate

(i) $x^3+2\sqrt{x}$ (ii) $(x+2)\ln x$.

(b) (i) Find from first principles the derivative of x^3 with respect to x.

(ii) Let $f(x)=\sin^4 x+\cos^4 x$.

Find the derivative of $f(x)$ and express it in the form $k\sin px$, where $k, p\in\mathbf{Z}$.

(c) If $\sin y=\frac{1}{2}(1-x^2)$ for $-\sqrt{3}<x<\sqrt{3}$,

calculate the value of a and the value of b when

$$\left(\frac{dy}{dx}\right)^2=\frac{a}{3-x^2}-\frac{b}{1+x^2},\ a,b\in\mathbf{N}_0.$$

SOLUTION

6 (a) (i)

$$y=x^3+2\sqrt{x}=x^3+2x^{\frac{1}{2}}$$

$$\boxed{y=x^n\Rightarrow\frac{dy}{dx}=nx^{n-1}}\ \ldots\ldots\ 1$$

$$\Rightarrow\frac{dy}{dx}=3x^2+2\times\tfrac{1}{2}x^{-\frac{1}{2}}=3x^2+x^{-\frac{1}{2}}$$

$$\therefore\frac{dy}{dx}=3x^2+\frac{1}{x^{\frac{1}{2}}}=3x^2+\frac{1}{\sqrt{x}}$$

6 (a) (ii)

$$y=(x+2)\ln x$$

$$\Rightarrow\frac{dy}{dx}=(x+2)(\tfrac{1}{x})+(\ln x)(1)$$

$$\therefore\frac{dy}{dx}=\frac{x+2}{2}+\ln x$$

$$\boxed{\begin{aligned}u&=(x+2)\Rightarrow\frac{du}{dx}=1\\ v&=\ln x\Rightarrow\frac{dv}{dx}=\frac{1}{x}\end{aligned}}$$

$$\boxed{\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}}\ \ldots\ldots\ 3$$

$$\boxed{y=\ln x\Rightarrow\frac{dy}{dx}=\frac{1}{x}}\ \ldots\ldots\ 8$$

6 (b) (i)

FIRST PRINCIPLES PROOF. If $y=x^3\Rightarrow\frac{dy}{dx}=3x^2$.

PROOF

$$y+\Delta y=(x+\Delta x)^3=x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3$$

$$y=x^3$$

$$\Delta y=3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3 \quad \text{by subtraction}$$

$$\therefore\frac{\Delta y}{\Delta x}=3x^2+3x\Delta x+(\Delta x)^2\Rightarrow\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=3x^2$$

6 (b) (ii)

$$y = f(x) = \sin^4 x + \cos^4 x = (\sin x)^4 + (\cos x)^4$$

$\boxed{y = \sin x \Rightarrow \dfrac{dy}{dx} = \cos x}$ **5**

$$\Rightarrow \frac{dy}{dx} = 4(\sin x)^3(\cos x) + 4(\cos x)^3(-\sin x)$$

$\boxed{y = \cos x \Rightarrow \dfrac{dy}{dx} = -\sin x}$ **6**

$$\Rightarrow \frac{dy}{dx} = 4\sin^3 x\cos x - 4\cos^3 x\sin x$$

$$\Rightarrow \frac{dy}{dx} = 4\sin x\cos x(\sin^2 x - \cos^2 x)$$

$\boxed{\sin 2A = 2\sin A\cos A}$ **13**

$$\Rightarrow \frac{dy}{dx} = 2\sin 2x(-\cos 2x)$$

$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$ **14**

$$\Rightarrow \frac{dy}{dx} = -2\sin 2x\cos 2x$$

$$\therefore \frac{dy}{dx} = -\sin 4x$$

6 (c)

$$\sin y = \tfrac{1}{2}(1-x^2) \Rightarrow y = \sin^{-1}[\tfrac{1}{2}(1-x^2)]$$

$\boxed{y = \sin^{-1} f(x) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-f(x)^2}} \times f'(x)}$ **9**

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-[\frac{1}{2}(1-x^2)]^2}} \times \tfrac{1}{2}(-2x)$$

$$\therefore \frac{dy}{dx} = -\frac{x}{\sqrt{1-[\frac{1}{2}(1-x^2)]^2}}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1-[\frac{1}{2}(1-x^2)]^2} = \frac{x^2}{1-\frac{1}{4}(1-x^2)^2}$$ [Multiply above and below by 4.]

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{4x^2}{4-(1-x^2)^2} = \frac{4x^2}{4-(1-2x^2+x^4)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{4x^2}{4-1+2x^2-x^4} = \frac{4x^2}{3+2x^2-x^4}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{4x^2}{(3-x^2)(1+x^2)}$$

$$\frac{4x^2}{(3-x^2)(1+x^2)} = \frac{a}{3-x^2} - \frac{b}{1+x^2}$$

$$\Rightarrow \frac{4x^2}{(3-x^2)(1+x^2)} = \frac{a(1+x^2)-b(3-x^2)}{(3-x^2)(1+x^2)}$$

$$\therefore 4x^2 = a(1+x^2) - b(3-x^2)$$

$$\therefore 4x^2 = a + ax^2 - 3b + bx^2$$

$$\therefore 4x^2 = (a-3b) + (a+b)x^2$$ [This is an identity.]

$$\therefore a - 3b = 0 \Rightarrow a = 3b$$

$$\therefore a + b = 4 \Rightarrow 3b + b = 4 \Rightarrow 4b = 4$$

$$\therefore b = 1,\ a = 3$$

7 (a) Take $x_1 = 3$ as the first approximation of a real root of the equation

$$x^3 - 6x^2 + 24 = 0.$$

Find, using the Newton-Raphson method, x_2, the second approximation and write your answer as a fraction.

(b) (i) Find the equation of the tangent to the curve

$$2x^2 - 3y^2 = 6$$

at the point $(-3, -2)$.

(ii) If $x = \dfrac{1-t^2}{1+t^2}$ and $y = \dfrac{2t}{1+t^2}$, find, as a fraction, the value of $\dfrac{dy}{dx}$ when $t = \frac{3}{4}$.

(c) Let $y = x - 1 + \dfrac{1}{x-1}$, $x \in \mathbf{R}$, $x \neq 1$.

(i) Find the values of x for which $\dfrac{dy}{dx} = 0$.

(ii) For x real, show that y cannot have a real value between -2 and $+2$.

SOLUTION

7 (a)

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad \ldots\ldots\ \mathbf{16}$$

STEPS
1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

1. $f(x) = x^3 - 6x^2 + 24$

2. $f'(x) = 3x^2 - 12x$

3. $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \Rightarrow x_2 = 3 - \dfrac{(3)^3 - 6(3)^2 + 24}{3(3)^2 - 12(3)}$

$$\Rightarrow x_2 = 3 - \frac{27 - 54 + 24}{27 - 36} = 3 - \frac{-3}{-9}$$

$$\Rightarrow x_2 = 3 - \frac{1}{3}$$

$$\therefore x_2 = \tfrac{8}{3}$$

7 (b) (i)

$$2x^2 - 3y^2 = 6$$

$$\Rightarrow 4x - 6y\frac{dy}{dx} = 0$$

$$\Rightarrow 4x = 6y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x}{6y} = \frac{2x}{3y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(-3,\,-2)} = \frac{2(-3)}{3(-2)} = 1$$

Equation of tangent:

$m = 1$, point $(-3, -2)$

$$x - y + k = 0$$

$$\Rightarrow (-3) - (-2) + k = 0$$

$$\Rightarrow -3 + 2 + k = 0$$

$$\therefore k = 1$$

Equation of tangent: $x - y + 1 = 0$

7 (b) (ii)

Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$$\frac{dy}{dt} = \frac{(1+t^2)2 - 2t(2t)}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dt} = \frac{2-2t^2}{(1+t^2)^2}$$

$$u = 2t \Rightarrow \frac{du}{dt} = 2$$

$$v = 1+t^2 \Rightarrow \frac{dv}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ 4

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2} = \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}$$

$$\therefore \frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

$$u = 1-t^2 \Rightarrow \frac{du}{dt} = -2t$$

$$v = 1+t^2 \Rightarrow \frac{dv}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2-2t^2}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}} = \frac{2-2t^2}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2-2t^2}{-4t} = \frac{t^2-1}{2t}$$

$$\therefore \left(\frac{dy}{dx}\right)_{t=\frac{3}{4}} = \frac{(\frac{3}{4})^2-1}{2(\frac{3}{4})} = \frac{\frac{9}{16}-1}{\frac{3}{2}} = \frac{-\frac{7}{16}}{\frac{3}{2}} = -\frac{7}{24}$$

7 (c) (i)

$$y = x-1+\frac{1}{x-1} = x-1+(x-1)^{-1}$$

$$\Rightarrow \frac{dy}{dx} = 1-1(x-1)^{-2}(1) = 1-\frac{1}{(x-1)^2}$$

$$y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$$ 1

$$\frac{dy}{dx} = 0 \Rightarrow 1-\frac{1}{(x-1)^2} = 0$$

$$\Rightarrow 1 = \frac{1}{(x-1)^2}$$

$$\Rightarrow (x-1)^2 = 1$$

$$\Rightarrow (x-1) = \pm 1$$

$$\therefore x = 0,\ 2$$

7 (c) (ii)

$y = x - 1 + \dfrac{1}{x-1}$ [Multiply across by $(x-1)$.]

$\Rightarrow (x-1)y = (x-1)(x-1) + 1$

$\Rightarrow xy - y = x^2 - 2x + 1 + 1$

$\Rightarrow 0 = x^2 - 2x - xy + y + 2$

$\Rightarrow 0 = x^2 - (y+2)x + (y+2)$

Remember: If $b^2 - 4ac \geq 0 \Rightarrow$ Real roots.
If $b^2 - 4ac < 0 \Rightarrow$ Unreal or complex roots.

$a = 1$
$b = -(y+2)$
$c = (y+2)$

$b^2 - 4ac < 0$

$\Rightarrow (y+2)^2 - 4(1)(y+2) < 0$

$\Rightarrow (y+2)[(y+2) - 4] < 0$

$\Rightarrow (y+2)(y-2) < 0$

Solve $(y+2)(y-2) = 0 \Rightarrow y = -2,\ 2$

	$\boxed{-2}$		$\boxed{2}$	
-3		0		3
$(-1)(-5) < 0$		$(2)(-2) < 0$		$(5)(1) < 0$
$5 < 0$		$-4 < 0$		$5 < 0$
Wrong		Correct		Wrong

Therefore, y cannot have a real value between –2 and 2.