1996

6 (a) Differentiate

(i) $\frac{2x}{x+1}$ (ii) $4e^{2x+1}$

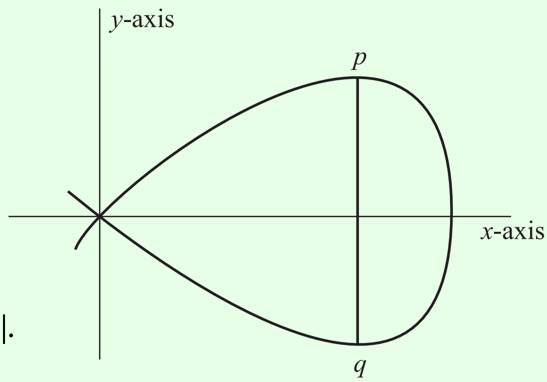
(b) (i) Find $\frac{dy}{dx}$ if $y = \ln\sqrt{x^2+1}$.

(ii) Take $x_1 = 1$ as the first approximation of a real root of the equation $x^3 - 2 = 0$. Find, using the Newton-Raphson method, x_2 and x_3 the second and third approximations. Write your answers as fractions.

(c) (i) $x = a(\theta + \sin\theta)$; $y = a(1-\cos\theta)$ where a is a constant.
Show

$$1+\left(\frac{dy}{dx}\right)^2 = \sec^2\left(\tfrac{\theta}{2}\right).$$

(ii) $[pq]$ is a chord of the loop of the curve $y^2 = x^2(6-x)$ so that the chord is parallel to the y-axis.
Calculate the maximum value of $|pq|$.



SOLUTION

6 (a) (i)

$$y = \frac{2x}{x+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x+1)2-2x(1)}{(x+1)^2} = \frac{2x+2-2x}{(x+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2}{(x+1)^2}$$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$v = x+1 \Rightarrow \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ 4

6 (a) (ii)

$$y = 4e^{2x+1}$$

$$\Rightarrow \frac{dy}{dx} = 4[e^{2x+1} \times 2]$$

$$\therefore \frac{dy}{dx} = 8e^{2x+1}$$

$$y = e^{f(x)} \Rightarrow \frac{dy}{dx} = e^{f(x)} \times f'(x)$$ 7

REMEMBER IT AS:

Repeat the whole function × Differentiation of the power.

6 (b) (i)

$y = \ln\sqrt{x^2+1} = \ln(x^2+1)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+1)$ [Use log rule No. 3]

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x^2+1}\times 2x\right]$

$\therefore \frac{dy}{dx} = \frac{x}{x^2+1}$

LOG RULES

3. $N\log_a M = \log_a(M^N)$

$y = \ln f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)}\times f'(x)$ **8**

REMEMBER IT AS:

One over the function inside the log × Differentiation of function inside the log

6 (b) (ii)

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ **16**

STEPS

1. Write down $f(x)$.
2. Do $f'(x)$.
3. Substitute starting value x_n into formula **16**.
4. Repeat if asked.

1. $f(x) = x^3 - 2$

2. $f'(x) = 3x^2$

3. $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)}$

$\Rightarrow x_2 = 1 - \frac{(1)^3-2}{3(1)^2} = 1 - \frac{1-2}{3}$

$\therefore x_2 = 1 - \frac{(-1)}{3} = 1 + \frac{1}{3} = \frac{4}{3}$

4. $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{4}{3} - \frac{f(\frac{4}{3})}{f'(\frac{4}{3})}$

$\Rightarrow x_3 = \frac{4}{3} - \frac{(\frac{4}{3})^3 - 2}{3(\frac{4}{3})^2} = \frac{91}{72}$

6 (c) (i)

Do $\frac{dy}{dt}$ first, then do $\frac{dx}{dt}$, and then divide $\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx}$

$y = a(1-\cos\theta) = a - a\cos\theta$

$\therefore \frac{dy}{d\theta} = a\sin\theta$

$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$ **5**

$x = a(\theta + \sin\theta) = a\theta + a\sin\theta$

$\therefore \frac{dx}{d\theta} = a + a\cos\theta$

$y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$ **6**

$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a + a\cos\theta} = \frac{\sin\theta}{1+\cos\theta}$

STEPS

1. You have to prove that the left-hand side (*LHS*) equals the right-hand side (*RHS*).
2. Change everything to sine and cosine.
3. Simplify each side using page 9 of the tables and good algebra.
4. For half angles, $\frac{\theta}{2}$: Let $\frac{\theta}{2} = A \Rightarrow \theta = 2A$.

Let $\frac{\theta}{2} = A \Rightarrow \theta = 2A$

LHS

$$1+\left(\frac{\sin\theta}{1+\cos\theta}\right)^2$$

$$=1+\left(\frac{\sin 2A}{1+\cos 2A}\right)^2 = 1+\left(\frac{2\sin A\cos A}{1+\cos^2 A-\sin^2 A}\right)^2$$

$$=1+\left(\frac{2\sin A\cos A}{2\cos^2 A}\right)^2$$

$$=1+\left(\frac{\sin A}{\cos A}\right)^2$$

$$=1+\frac{\sin^2 A}{\cos^2 A} = \frac{\cos^2 A+\sin^2 A}{\cos^2 A}$$

$$=\frac{1}{\cos^2 A}$$

RHS

$$\sec^2\left(\tfrac{\theta}{2}\right)$$

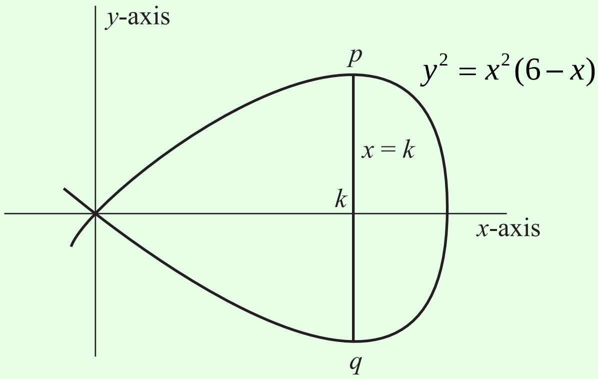
$$=\sec^2 A$$

$$=\frac{1}{\cos^2 A}$$

$\sec A = \dfrac{1}{\cos A}$

6 (c) (ii)

The line *pq* passes through a points on the *x*-axis, call it (*k*, 0). Therefore, the equation of *pq* is *x* = *k*. To find *p* and *q* solve simultaneously the equations of the line and the curve.



$$x = k \Rightarrow y^2 = k^2(6-k)$$

$$\therefore y = \pm k\sqrt{6-k}$$

$$\therefore p(k, k\sqrt{6-k}),\ q(k, -k\sqrt{6-k})$$

$$\therefore |pq| = D = 2k\sqrt{6-k}$$

You need to maximise the distance function, *D*, with respect to *k*.

$$\frac{dD}{dk} = 0 \Rightarrow (2k)(-\tfrac{1}{2}(6-k)^{-\frac{1}{2}}) + (6-k)^{\frac{1}{2}}(2) = 0$$

$$\Rightarrow (6-k)^{\frac{1}{2}}(2) = \frac{k}{(6-k)^{\frac{1}{2}}}$$

$$\Rightarrow 2(6-k) = k$$

$$\Rightarrow 12-2k = k \Rightarrow 12 = 3k$$

$$\therefore k = 4$$

$$\therefore D_{\text{Max.}} = 2(4)\sqrt{6-4} = 8\sqrt{2}$$

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ **3**

$$u = 2k \Rightarrow \frac{du}{dx} = 2$$

$$v = (6-k)^{\frac{1}{2}} \Rightarrow \frac{dv}{dx} = \tfrac{1}{2}(6-k)^{-\frac{1}{2}}(-1)$$

7 (a) Find from first principles the derivative of x^2 with respect to x.

(b) The function f is defined

$$f : x \to (x-4)\{(x-3)^2+4\}.$$

Find

(i) $f(3)$

(ii) the derivative with respect to x of the function at $x = 3$.

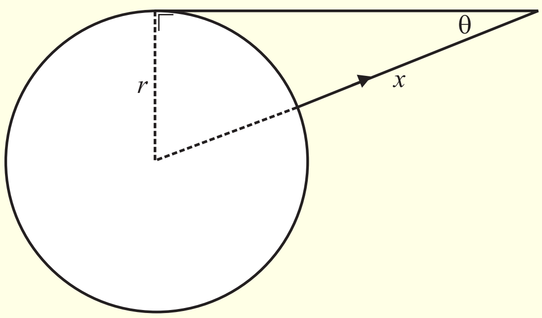
(iii) the equation of the tangent at $(3, f(3))$.

Show that the tangent and the graph of $x \to f(x)$ will both intersect the x-axis at the same point.

(c) (i) Given $\tan y = x$, show $\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$ and hence, find $\frac{d}{dx}\tan^{-1} x$.

(ii) An astronaut is at a height x km above the earth, as shown.
He moves vertically away from the earth's surface at a velocity $\frac{dx}{dt}$ of $\frac{r}{5}$ km/h where r is the length of the earth's radius.
He observes the angle θ as shown.
Express x in terms of r and θ.

Hence find $\frac{d\theta}{dt}$ when $x = r$.



SOLUTION

7 (a)

FIRST PRINCIPLES PROOF. If $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$.

PROOF

$y + \Delta y = (x+\Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$

$y = x^2$

$\Delta y = 2x\Delta x + (\Delta x)^2$ by subtraction

$$\therefore \frac{\Delta y}{\Delta x} = 2x + \Delta x \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

7 (b) (i)

$f(x) = (x-4)\{(x-3)^2+4\}$

$\Rightarrow f(3) = (3-4)\{(3-3)^2+4\}$

$\therefore f(3) = (-1)\{4\} = -4$

7 (b) (ii)

$f(x) = (x-4)\{(x-3)^2 + 4\}$

$\Rightarrow f'(x) = (x-4)2(x-3)^1(1) + \{(x-3)^2 + 4\}(1)$

$\Rightarrow f'(x) = 2(x-4)(x-3) + \{(x-3)^2 + 4\}$

$\therefore f'(3) = 2(3-4)(3-3) + \{(3-3)^2 + 4\}$

$\Rightarrow f'(3) = 2(-1)(0) + \{(0)^2 + 4\} = 0 + 4$

$\therefore f'(3) = 4$

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ **3**

$u = (x-4) \Rightarrow \dfrac{du}{dx} = 1$

$v = \{(x-3)^2 + 4\} \Rightarrow \dfrac{dv}{dx} = 2(x-3)$

7 (b) (iii)

Equation of the tangent *T*: *m* = 4, point (3, 4)

$\therefore T: 4x - y + k = 0$

$(3, -4) \in T \Rightarrow 4(3) - (-4) + k = 0$

$\Rightarrow 12 + 4 + k = 0$

$\Rightarrow k = -16$

$\therefore T: 4x - y - 16 = 0$

Intersect *x*-axis: Put *y* = *f* (*x*) = 0.

$T: y = 0 \Rightarrow 4x - 0 - 16 = 0$

$\Rightarrow 4x = 16$

$\therefore x = 4 \Rightarrow (4,\ 0)$ is the *x*-intercept.

$f(x) = 0 \Rightarrow (x-4)\{(x-3)^2 + 4\} = 0$

$\Rightarrow (x-4) = 0$

$\therefore x = 4 \Rightarrow (4,\ 0)$ is one of the *x*-intercepts.

7 (c) (i)

$\tan y = x$

$y = \tan x \Rightarrow \dfrac{dy}{dx} = \sec^2 x$

$\Rightarrow \sec^2 y \times \dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1 + \tan^2 y}$

$\sec^2 A = 1 + \tan^2 A = \dfrac{1}{\cos^2 A}$

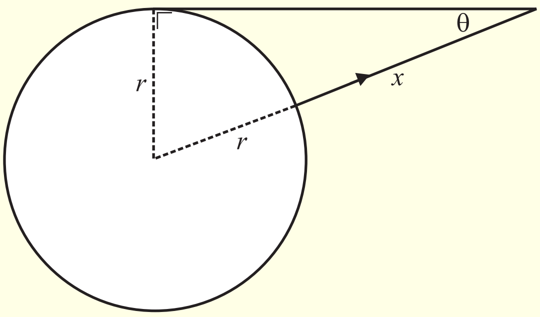
$\tan y = x \Rightarrow y = \tan^{-1} x$

$\dfrac{d}{dx}(y) = \dfrac{1}{1 + \tan^2 y}$

$\Rightarrow \dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$

7 (c) (ii)

Steps
1. Write down rate of changes given.
2. Write down the rate of change to be found.
3. Write down a formula involving the non-time variables.
4. Differentiate this formula implicitly with respect to time (t).
5. Substitute in the numbers.



1. $\frac{dx}{dt} = \frac{r}{5}$

2. $\frac{d\theta}{dt} = ?$

3. $\sin\theta = \frac{r}{x+r} \Rightarrow x + r = \frac{r}{\sin\theta}$

$\Rightarrow x = \frac{r}{\sin\theta} - r$

$\Rightarrow x = r(\sin\theta)^{-1} - r$

4. $\frac{dx}{dt} = -r(\sin\theta)^{-2}(\cos\theta) \times \frac{d\theta}{dt} = -\frac{r\cos\theta}{\sin^2\theta} \times \frac{d\theta}{dt}$

$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$ **5**

5. $\frac{r}{5} = -\frac{r\cos\theta}{\sin^2\theta} \times \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{\sin^2\theta}{5\cos\theta}$

$x = r : \sin\theta = \frac{r}{r+r} = \frac{1}{2} \Rightarrow \theta = 30^{\circ}$

$\therefore \frac{d\theta}{dt} = -\frac{\sin^2 30^{\circ}}{5\cos 30^{\circ}} = -\frac{(\frac{1}{2})^2}{5(\frac{\sqrt{3}}{2})} = -\frac{\frac{1}{4}}{\frac{5\sqrt{3}}{2}} = -\frac{1}{4} \times \frac{2}{5\sqrt{3}}$

$\therefore \frac{d\theta}{dt} = -\frac{1}{10\sqrt{3}}$